1 Continuous Uniform Distribution

A r.v. X is said to be uniformly distributed over the interval $[\alpha, \beta]$ if its probability distribution is given by
$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$
The uniform distribution arises in practice when we suppose a certain r.v is equally likely to be near any value in the interval $[\alpha, \beta]$.
If $[a, b] \subset [\alpha, \beta]$ then

$$P(a \le X \le b) = \int_a^b f(x)dx = \frac{1}{\beta-\alpha}[x]_a^b$$
$$= \frac{b-a}{\beta-\alpha}$$

Example 1.1. If X is uniformly distributed over the integral 0 to 10 then

- $P(2 < X < 9) = \frac{9-2}{10-0} = \frac{7}{10}$
- $P(1 \le X < 4) = \frac{3}{10}$
- $P(0 \le X < 5) = \frac{5}{10}$

Example 1.2. Buses arrive at a specified stop at 15 min intervals starting at 7 am. If a passenger arrive at a stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

1. less than 5 minutes for a bus
2. at least 12 minutes for a bus

Solution

1. Let X denotes the time in minutes past 7 am that the passenger arrives at the stop.
   Since X is uniform r.v over $[0, 30]$, it follows that passenger will have to wait less than 5 minutes if he arrives between 7:10 and 7:15 or 7:25 and 7:30.
   Hence the required probability is $P(10 < X < 15) + P(25 < X < 30) = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$.
2. Similarly the passenger will have to wait at least 12 minutes if he arrives after 7:00 and before 7:03 or after 7:15 and before 7:18.
   Hence the required probability is $P(0 < X < 3) + P(15 < X < 18) = \frac{3}{30} + \frac{3}{30} = \frac{1}{5}$.

Expectation:

$$E(X) = \int_\alpha^\beta x f(x)dx$$
$$= \int_\alpha^\beta \frac{x}{\beta-\alpha}dx$$
$$= \frac{1}{\beta-\alpha}[\frac{x^2}{2}]_\alpha^\beta$$
$$= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$$

$$
\begin{aligned}
\therefore E(X) &= \frac{\alpha+\beta}{2} \\
E(X^2) &= \int_{\alpha}^{\beta} x^2 f(x)dx \\
&= \frac{1}{\beta-\alpha}[\frac{x^3}{3}]_{\alpha}^{\beta} \\
&= \frac{1}{3(\beta-\alpha)}(\frac{\beta^3-\alpha^3}{3}) \\
&= \frac{1}{3}(\beta^2+\beta\alpha+\alpha^2)
\end{aligned}
$$

Variance:

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= \frac{1}{3}(\beta^2+\beta\alpha+\alpha^2) - (\frac{\alpha+\beta}{2})^2 \\
&= \frac{1}{3}(\beta^2+\beta\alpha+\alpha^2) - \frac{\alpha^2+\beta^2+2\alpha\beta}{4} \\
&= \frac{1}{12}[4\beta^2+4\beta\alpha+4\alpha^2-3\alpha^2-3\beta^2-6\alpha\beta] \\
&= \frac{1}{12}[\alpha-\beta]^2
\end{aligned}
$$

M.G.F:

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_{\alpha}^{\beta} e^{tx} f(x)dx \\
&= \frac{1}{\beta-\alpha}[\frac{e^{tx}}{t}]_{\alpha}^{\beta} \\
&= \frac{1}{\beta-\alpha}[\frac{e^{t\beta}-e^{t\alpha}}{t}] \\
&= \frac{1}{t(\beta-\alpha)}[e^{t\beta}-e^{t\alpha}]
\end{aligned}
$$

Distribution function

$$
\begin{aligned}
F(x) &= \int_{\alpha}^{x} f(t)dt \\
&= \frac{1}{\beta-\alpha}(t)_{\alpha}^{x} \\
&= \frac{x-\alpha}{\beta-\alpha}
\end{aligned}
$$

2 Exponential Distribution

This specifies the time until the next event. It is continuous distribution analogous to the geometric distribution for a Bernoulli trial.

Definition 2.1. A continuous r.v is called exponential if its p.d.f. is given by

$$
f(x) = \begin{cases} \lambda e^t x & x \geq 0 \\ 0 & x < 0 \end{cases}
$$

Here $\lambda > 0$ is the parameter and is denoted by $x \sim exp(\lambda)$

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M.G.F

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)dx \\ &= \int_{0}^{\infty} \lambda e^{-\lambda x} e^{tx} dx \\ &= \lambda \int_{0}^{\infty} e^{(-\lambda+t)x} dx \\ &= \lambda [\frac{e^{(t-\lambda)x}}{t-\lambda}]_0^{\infty} \\ &= \frac{\lambda}{t-\lambda}[0-1], \text{if} \qquad t<\lambda \\ &= \frac{\lambda}{t-\lambda}, \text{when} \qquad (t<\lambda) \end{aligned}$$

Expectation

$$\begin{aligned} E(X) = M'_X(0) &= [\lambda \frac{-1}{(\lambda-t)^2}(-1)]_{t=0} \\ &= [\frac{\lambda}{(\lambda-t)^2})]_{t=0} \\ &= \frac{\lambda}{\lambda^2} \\ &= \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} E(X^2) = M''_X(0) &= [\frac{\lambda}{(\lambda-t)^2}]^1_{t=0} \\ &= [\frac{(-2)\lambda(-1)}{(\lambda-t)^3}]_{t=0} \\ &= [\frac{2\lambda}{(\lambda-t)^3}]_{t=0} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Variance

$$\begin{aligned} Var(X) &= E(X^2) - E^2(X) \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \\ &= (\frac{1}{\lambda})^2 \\ &= (mean)^2 \end{aligned}$$

Distribution function:

$$\begin{aligned} F(k) &= P(X \leq k) \\ &= \int_0^k f(x)dx \\ &= \int_0^k \lambda e^{-\lambda x} dx \end{aligned}$$

$$= \lambda[\frac{e^{-\lambda x}}{-\lambda}]_0^k$$
$$= 1 - e^{-\lambda k}$$

Theorem 2.2. ***(Memory loss property)***
If $X \sim exp(\lambda)$ then $P[X > t + s|X > t] = P[X > s]$ $\forall t, s \geq 0$.

Proof. $P[X < k] = 1 - P[X \leq k] = 1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$

$$\begin{aligned} P[X > t + s|X > t] &= \frac{P[X > t + s, X > t]}{P[X > t]} \\ &= \frac{P[X > t + s]}{P[X > t]} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} \\ &= P[X > s] \end{aligned}$$

□

Remark 2.3. A continuous r.v. X satisfies memory loss property iff it is exponential distribution.

3 Gamma distribution

This is similar to negative binomial distribution. It specifies the time until the α^{th} event, so it is analogous to the negative binomial distribution.

Definition 3.1. A continuous r.v. is said to have gamma distribution with parameters (α, λ) $\lambda > 0, \alpha > 0$ if its density function is given by
$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$
Where

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty \lambda e^-\lambda x(\lambda x)^{\alpha-1}dx \\ &= \int_0^\infty e^{-y}y^{\alpha-1}dy \end{aligned}$$

Remark 3.2. If $\alpha = 1$, Gamma distribution is same as exponential distribution.

M.G.F:

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx}f(x)dx \\ &= \int_0^\infty \frac{e^{tx}\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{(t-\lambda)}x^{\alpha-1}dx \end{aligned}$$

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$$
\begin{aligned}
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(-t+\lambda)} x^{\alpha-1} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-y} \frac{y^{\alpha-1}}{(\lambda-t)^{\alpha-1}(\lambda-t)} dy \\
&= \frac{\lambda^\alpha \Gamma(\alpha)}{\Gamma(\alpha)(\lambda-t)^\alpha} \text{ (Put}(\lambda-t)x = y\text{then}(\lambda-t)dx = dy \\
M_x(t) &= \frac{\lambda^\alpha}{(\lambda-t)^\alpha}
\end{aligned}
$$

Exercise: Derive $E(x)$ and $V(x)$ from M.G.F.

4 Beta Distribution

A r.v. X is said to follow Beta distribution with parameter α, β if its p.m.f. is given by
$f(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & 0 < x < 1, \alpha, \beta > 0; \\ 0, & \text{otherwise.} \end{cases}$
Where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ or $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

Remark 4.1. If $\alpha = \beta = 1$, then $X \sim U(0,1)$.

Expectation

$$
\begin{aligned}
E(X) &= \int_0^1 x f(x) dx \\
&= \int_0^1 \frac{x x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+1,\beta) \\
&= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
E(x) &= \frac{\alpha}{\alpha+\beta} \quad (\because \Gamma(k+1) = k\Gamma k)
\end{aligned}
$$

Exercise: $E(x^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$.

$$
\begin{aligned}
Var(x) &= E(x^2) - E^2(x) \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - (\frac{\alpha}{\alpha+\beta})^2 \\
&= \frac{(\alpha^2+\alpha)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
Var(x) &= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.
\end{aligned}
$$

5 Normal distribution

Definition 5.1 (Normal distribution). A random variable is said to be normally distributed with parameterμ and σ^2 if its p.d.f is given by

$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$
and is denoted by $X \sim N(\mu, \sigma^2)$.

Definition 5.2. If a r.v. X follows $N(0,1)$ then X is called standard normal distribution.i.e. a random variable X is said to follow standard normal distribution if its pdf is given by
$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad -\infty < x < \infty$

Remark 5.3. Since $\Phi(x)$ is a p.d.f, we have $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = 1$.

Expectation:

$$E(x-\mu) = \int_{-\infty}^{\infty} (x-\mu) f(x) dx$$
$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let $x - \mu = y \Rightarrow E(X-\mu) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{\frac{-y^2}{2\sigma^2}} dy$
Since $ye^{\frac{-y^2}{2}}$ is an odd function, we get

$$\int_{-\infty}^{\infty} y e^{\frac{-y^2}{2}} dy = 0$$
$$E(x-\mu) = 0$$
$$E(X) = \mu$$

Variance:

$$Var(x) = E((x-\mu)^2)$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = y$. Then

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 y^2 e^{\frac{-y^2}{2}} \sigma dy$$
$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y (y e^{\frac{-y^2}{2}} dy)$$
$$[= \frac{\sigma^2}{\sqrt{2\pi}} y \int_{-\infty}^{\infty} (y e^{\frac{-y^2}{2}} dy)]$$
$$= \frac{\sigma^2}{\sqrt{2\pi}} [[-y e^{\frac{-y^2}{2}}]_{-\infty}^{\infty} - \int (-) e^{\frac{-y^2}{2}} dy)]$$
$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy$$
$$= \sigma^2 .1$$
$$= \sigma^2$$

Theorem 5.4. *If X is normal with mean μ and variance σ^2 then for any constant a and non zero b the random variable $y = a + bX$ is also a normal r.v. with mean $a + b\mu$ and variance $b^2\sigma^2$ (i.e $X\sim (\mu, \sigma^2)$ then $a + bX \sim N(a + b\mu, b^2\sigma^2)$).*

Proof. For $b > 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(a + bX \leq y) \\ &= P(X \leq \frac{y-a}{b}) \\ &= F_X(\frac{y-a}{b}) \end{aligned}$$

For $b < 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(a + bX \leq y) \\ &= P(bX \leq y - a) \\ &= P(X \geq \frac{y-a}{b}) \\ &= 1 - F_X(\frac{y-a}{b}) \end{aligned}$$

$F_Y(y) = \begin{cases} F_X(\frac{y-a}{b}), & b > 0; \\ 1 - F_X(\frac{y-a}{b}), & b < 0; \end{cases}$

$\therefore$ by differentiating the $F_Y(y)$, we get p.d.f of Y as

$$\begin{aligned} f_Y(y) &= \begin{cases} \frac{1}{b} f_X(\frac{y-a}{b}), & b > 0; \\ \frac{-1}{b} f_X(\frac{y-a}{b}), & b > 0. \end{cases} \\ &= \frac{1}{|b|} f_X(\frac{y-a}{b}) \\ &= \frac{1}{|b|} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\frac{y-a}{b}-\mu)^2}{2\sigma^2}} \\ &= \frac{1}{|b|} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-(a+b\mu))^2}{2b^2\sigma^2}} \\ f_Y(y) &= \frac{1}{(\sigma|b|)\sqrt{2\pi}} e^{\frac{-(y-(a+b\mu))^2}{2b^2\sigma^2}} \end{aligned}$$

$Y \sim N(a + b\mu, b^2\sigma^2)$. □

Remark 5.5. $X \sim N(\mu, \sigma^2)$ and $\frac{X-\mu}{\sigma} = Z$. If $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} = Z \sim N(0, 1)$

Proof. Let $Z = \frac{X-\mu}{\sigma} \Rightarrow Z = \frac{-\mu}{\sigma} + \frac{1}{\sigma}X$

Here $a = \frac{-\mu}{\sigma}$, $b = \frac{1}{\sigma}$

$a + b\mu = \frac{-\mu}{\sigma} + \frac{\mu}{\sigma}$ and $b^2\sigma^2 = \frac{1}{\sigma^2}\sigma^2 = 1$

$Z \sim N(a + b\mu, b^2\sigma^2) \Rightarrow Z \sim N(0, 1)$

$\therefore Z$ is standard normal r.v. and it's distribution function is given by

$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy$

□

$$P(X \leq b) = p(\frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma})$$

$$
\begin{aligned}
&= P(z < \frac{b-\mu}{\sigma}) \\
&= \Phi(\frac{b-\mu}{\sigma}) \\
P(a < X < b) &= P(a-\mu < x-\mu < b-\mu) \\
&= P(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}) \\
&= P(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}) \\
&= \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})
\end{aligned}
$$

$$
\begin{aligned}
\Phi(-x) &= P(z < -x) \\
&= 1 - P(z < x) \\
\Phi(-x) &= 1 - \Phi(x)
\end{aligned}
$$

Example 5.6. If X is a normal random variable with mean $\mu = 3$ and variance $\sigma^2 = 16$, find $P(X < 11)$, $P(X > -1)$ and $P(2 < X < 7)$.

Solution:

$$
\begin{aligned}
P(X < 11) &= (\frac{X-\mu}{\sigma} < \frac{11-\mu}{\sigma}) \\
&= P(Z < \frac{11-3}{4}) \\
&= P(Z < \frac{8}{4}) \\
&= P(Z < 2) \\
&= \Phi(2) \\
&= 0.9772 \\
P(X > -1) &= 1 - P(X < -1) \\
&= 1 - P(\frac{X-\mu}{\sigma} < \frac{-1-\mu}{\sigma}) \\
&= 1 - P(Z < \frac{-4}{4}) \\
&= 1 - \Phi(-1) \\
&= X - 1 + \Phi(1) \\
&= \Phi(1) \quad \text{since} \quad (\Phi(-1) = 1 - \Phi(1)) \\
&= 0.8413 \\
P(2 < X < 7) &= P(\frac{2-\mu}{\sigma} < Z < \frac{7-\mu}{\sigma}) \\
&= P(\frac{-1}{4} < Z < \frac{4}{4}) \\
&= P(\frac{-1}{4} < Z < 1) \\
&= \Phi(1) - \Phi(\frac{-1}{4}) \\
&= \Phi(1) - [1 - \Phi(0.25)]
\end{aligned}
$$

$$= 0.8413 - 1 + 0.5987$$
$$= 1.4400 - 1$$
$$= 0.44$$

M.G.F.

First we find moment generating function for standard normal random variable Z

$$\begin{aligned}
M_Z(t) &= E(e^{tZ}) \\
&= \int_{-\infty}^{\infty} e^{tz} f(z)dz \\
&= \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-t)^2}{2}+\frac{t^2}{2}} dz \\
&= \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-t)^2}{2}} dz \\
&= \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(y)^2}{2}} dy \qquad (\because z-t=y) \\
&= e^{\frac{t^2}{2}}(1) \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

$\therefore$ For any normal random variable X

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= E(e^{t(\mu+\sigma z)}) \\
&= E(e^{t\mu} e^{t\sigma z}) \\
&= e^{t\mu} E(e^{t\sigma z}) \\
&= e^{t\mu}(e^{\frac{t^2\sigma^2}{2}}) \\
M_X(t) &= e^{t\mu+\frac{t^2\sigma^2}{2}}
\end{aligned}$$

Theorem 5.7. *The sum of independent normal random variable is again normal. i.e. if $X_i \sim N(\mu_i, \sigma_i^2)$ and X_i's are independent then*
$\sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i)$.

Proof. $E(e^{t\sum_{i=1}^n X_i}) = E(e^{tX_1} e^{tX_2} ....e^{tX_n})$
Since X_i's are independent and e^{tX} is continuous we get e^{tX_i} are also independent

$$\begin{aligned}
\Rightarrow E(e^{t\sum_{i=1}^n X_i}) &= E(e^{tX_1})E(e^{tX_2})\ldots E(e^{tX_n}) \\
&= (e^{t\mu_1+\frac{x^2\sigma_1^2}{2}})(e^{t\mu_2+\frac{x^2\sigma_2^2}{2}})\ldots(e^{t\mu_n+\frac{x^2\sigma_n^2}{2}}) \\
&= e^{t\sum_{i=1}^n \mu_i + \frac{t^2}{2}(\sum_{I=1}^n \sigma_i^2)}
\end{aligned}$$

$\sum X_i \sim N(\sum_{i=1}^n \mu_i^2, \sum_{i=1}^n \sigma_i^2)$. □

Example 5.8. 1. If X is normal that is $X \sim N(1,1)$ and $Y \sim N(2,4)$ then $X+Y \sim N(3,5)$

2. Suppose $X \sim N(1,1)$ and $A = -2 < x < 1$, $B = -1 < x < 1$, $c = 0 < x < 2$ then $P(B) \leq P(A)$.

Solution: $X \sim N(1,1)$ then $2 = \frac{X-\mu}{\sigma}$=$X-1 \sim N(0,1)$

and $A = \{-2 < x < 1\} = \{-2-1 < x-1 < 1-1\} \Rightarrow A = \{-3 < z < 0\}$,

$B = \{-1 < x < 1\} = \{-1-1 < z < 1-1\} \Rightarrow B = \{-2 < z < 0\}$

Similarly, $C = \{0 < x < 2\} = \{-1 < z < 1\}$

i.e. $B \subset A \Rightarrow P(B) \leq P(A)$.

6 Cauchy Distribution

A continuous random variable X is said to be follow cauchy distribution with parameters μ *and* σ if its pdf is given by

$f(x) = \frac{1}{\pi\sigma}\frac{1}{1+\frac{(x-\mu)^2}{\sigma^2}} \qquad -\infty < x < \infty, \qquad \sigma > 0, \qquad -\infty < \mu < \infty$

1. It is symmetric around μ.
2. Bell shape with peak flatter than that of normal distribution $N(\mu, \sigma^2)$.
3. $f(x)$ tends to zero as $x \to \pm\infty$ but the rate at which cauchy pdf tends to zero is much slower than the rate of which normal pdf goes to zero.i.e. The tails of cauchy distribution are much thicker than normal.

Definition 6.1. A r.v. whose pdf is given by

$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad -\infty < x < \infty$

is called the standard cauchy r.v. and is denoted by $X \sim C(0,1)$

Result Expectation and variance of cauchy random variable doesn't exists.

Proof.

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

$$\text{assume that} \quad X \sim C(0,1)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} x\frac{1}{1+x^2}dx$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} 2x\frac{1}{1+x^2}dx$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} [log(1+x^2]$$

$$= \infty$$

$\therefore E(x)$ is doesnt exist

□

Remark 6.2. μ and σ are just parameters they are not mean and variance. Similarly variance of X does not exist.

Distribution Function

$$\begin{aligned}
F(x) &= P(X \leq x) \qquad X \sim C(\mu, \sigma) \\
&= \int_{-\infty}^{x} f(t)dt \\
&= \int_{-\infty}^{x} \frac{1}{\pi\sigma} \frac{1}{(1 + (\frac{t-\mu}{\sigma})^2)} \\
\text{put} \quad \frac{t-\mu}{\sigma} &= y \\
dt &= \sigma dy \\
&= \frac{1}{\pi\sigma} \int_{-\infty}^{\frac{x-\mu}{\sigma}} x \frac{1}{1+y^2} \sigma dy \\
&= \frac{1}{\pi} \int_{-\infty}^{\frac{x-\mu}{\sigma}} x \frac{1}{1+y^2} dy \\
&= \frac{1}{\pi} [\tan^{-1}(y)]_{-\infty}^{\frac{x-\mu}{\sigma}} \\
&= \frac{1}{\pi} [\tan^{-1}(\frac{x-\mu}{\sigma}) - (\frac{-\pi}{2})] \\
&= \frac{1}{2} + \frac{1}{\bar{n}} [\tan^{-1}(\frac{x-\mu}{\sigma})].
\end{aligned}$$

7 NET/SET Questions

1. Molly earned a score of 940 on a national achievement test. The mean test score was 850 with a standard deviation of 100. What proportion of students had a higher score than Moly?(Assume that test scores are normally distributed):

   A. 0.10

   B. 0.50

   C. 0.18

   D. 0.82

   Solution:
   Given that $\mu = 850$ and $\sigma = 100$.
   Given $X \sim N(\mu, \sigma)$, we have to find $P(X > 940)$.

$$\begin{aligned}
P(X > 940) &= P(\frac{X - 850}{100} > \frac{90}{100}) \\
&= P(Z > 0.9) \\
&= 1 - \Phi(0.9) \\
&= 1 - 0.8159
\end{aligned}$$

$$= 0.18$$

Therefore, 0.18% of students had a higher score than Moly.

2. Let X be a normal r.v. with mean 1 and variance 1 and Y be a normal random variable with mean 2 and variance 4. Then $(X+Y)$ has

   A. mean 3 and variance 5

   B. mean 3 and variance less than equal to 9

   C. mean 3 and variance greater than 10

   D. mean greater than 3 and variance 5.

   Solution: If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then $(X+Y) \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.
   Hence $(X+Y)$ has mean 3 and variance 5.

3. Let Φ denote the distribution of standard normal variable. Which of the following statements is not always true?

   A. $\Phi(x) = \Phi(-x)$

   B. $\Phi(-x) = 1 - \Phi(x)$

   C. $\Phi(0) = 1/2$

   D. for $a > 0$, $\Phi(x+a) = \Phi(x) + \Phi(-x) - \Phi(-x-a)$.

   Solution: We know that $\Phi(-x) = 1 - \Phi(x)$.
   So, in particular, $\Phi(-0) = 1 - \Phi(0) \Rightarrow \Phi(0) = 1/2$
   Also $\Phi(x+a) = 1 - \Phi(-x-a)$ and hence $\Phi(x+a) = \Phi(x) + \Phi(-x) - \Phi(-x-a)$.
   $\Phi(x) = \Phi(-x)$ need not be true always.

4. I.Q. examination for 10th standard students are normally distributed with mean 100 and s.d. 14.2. Which of the following events is least likely to occur, if X denotes the I.Q. score of a randomly chosen 10th standard students?

   A. $X > 130$

   B. $X < 80$

   C. $90 < X < 115$

   D. $90 < X < 100$

   Solution: $P[X > 130] = P[\frac{X-100}{14.2} > \frac{130-100}{14.2}] = P[Z > 2.112] = 0.9826$
   $P[X < 80] = P[Z < -1.408] = 0.793$
   $P[90 < X < 115] = P[-0.7 < Z < 1.05] = \Phi(1.05) - \Phi(-0.7) = 0.8531 - 0.2420 = 0.61$
   $P[90 < X < 100] = P[-0.7 < Z < 0] = \Phi(0) - \Phi(-0.7) = 0.5 - 0.2420 = 0.258$
   Hence $90 < X < 100$ is least likely to occur.

5. Let X be a r.v. with $N(1,1)$. Define the events:
   $A_1 = \{-2 < X < 1\}$
   $B_1 = \{-1 < X < 1\}$
   $C_1 = \{0 < X < 2\}$
   Which of the following statement is correct?

   A. $P(B_1) < P(A_1) < P(C_1)$

   B. $P(C_1) < P(B_1) < P(A_1)$

   C. $P(B_1) = P(A_1) < P(C_1)$

   D. $P(B_1) = P(A_1) = P(C_1)$

   Solution: If $X \sim N(1,1)$, then $Z = X - 1 \sim N(0,1)$.
   Then $A_1 = \{-2 < X < 1\} = \{-3 < Z < 0\}$
   $B_1 = \{-1 < X < 1\} = \{-2 < Z < 0\}$
   $C_1 = \{0 < X < 2\} = \{-1 < Z < 1\}$
   Clearly $B_1 \subset A_1$. So $P(B_1) < P(A_1)$.

6. The amount of calories in a chocolate bar is normally distributed with an average of 250 calories. If 99.7% of all the bars have between 205 and 295 calories, then the standard deviation(in calories) is :

   A. 90

   B. 45

   C. 15

   D. 10.

   Solution: Since the 99.7% of all data lies within 3 standard deviations of the mean, we get that $\mu+3\sigma = 295$. Hence $\sigma = 15$.

7. Chebyshev's inequality can be applied to the distribution if its is

   (a) Any distribution regardless of its shape

   (b) Normal

   (c) Exponential

   (d) Gamma

   Solution:
   Chebyshev's inequality can be applied to any distribution regardless of its shape.

8. If characteristic function of X is $\phi(u)$, then the characteristic function of $b + cX$ is

   (a) $e^{ib}\phi(-uc)$

   (b) $e^{iu}\phi(ub)$

   (c) $e^{iub}\phi(uc)$

(d) $1 - e^{iub}\phi(uc)$

Solution:

Characteristic function of X is $\phi_X(u) = E(e^{iuX})$. Hence

$$\begin{aligned}\phi_{b+cX} &= E(e^{iu(b+cX)}) \\ &= E(e^{iub}e^{i(uc)X}) \\ &= e^{iub}E(e^{i(uc)X}) \\ &= e^{iub}\phi_X(u)\end{aligned}$$

9. One of the effects of flooding a lake is that mercury is leached from the soil, enters the food chain and contaminates the fish. Suppose that the concentration of mercury in an individual fish follows an approximate normal distribution with a mean of 0.25ppm and a standard deviation of 0.08 ppm. Fish are safe to eat if the mercury level is below 0.30ppm. What portion of fish would be safe to eat?

   (a) 23%

   (b) 63%

   (c) 73%

   (d) 27%

   Solution:

   We have to find $P(X < 0.30)$.

$$\begin{aligned}P(X < 0.30) &= P(\frac{X - 0.25}{0.08} < \frac{0.30 - 0.25}{0.08}) \\ &= P(Z < 0.625) \\ &= \Phi(0.625) \\ &= 0.7340\end{aligned}$$

   Hence 73% of fish are safe to eat.

10. Marks in a chemistry test follow a normal distribution with a mean of 65 and a standard deviation of 12. Approximately what percentage of the students would score below 50?

   (a) 22%

   (b) 11%

   (c) 15%

   (d) 18%

   Solution:

   We have to find $P(X < 50)$.

$$P(X < 50) = P(\frac{X - 65}{12} < \frac{-15}{12})$$

$$= P(Z < -1.25)$$
$$= 1 - \Phi(1.25)$$
$$= 1 - 0.8944$$
$$= 10.56$$

Hence approximately 11% of students score below 50.

11. Let X have a standard normal distribution. Then $P[X^2 > 0]$ is

    (a) 1/8

    (b) 1/4

    (c) 1/2

    (d) 1

    Solution:
    Since $X^2 > 0$ is the entire area under the curve, we get $P[X^2 > 0] = 1$

12. Let X have an exponential distribution with mean 4. Then $P[0 \leq X \leq 4]$ is

    (a) $= 1/2$

    (b) $> 1/2$

    (c) $> 2/3$

    (d) $\leq 1/2$

    Solution:
    $P[0 \leq X \leq 4] = F(4) - F(0)$ For an exponential X, $F(x) = 1 - e^{-\lambda x}$.
    Hence $P[0 \leq X \leq 4] = 1 - e^{-16} > 2/3$

13. Let the random variable X have a normal distribution with mean zero and variance 4. Then

    (a) $E[\cos x] = 2$

    (b) $E[\cos x] = e^{-2}$

    (c) $E[\cos x] = 1/2$

    (d) $E[\cos x] = e^{-6}$

    Solution:
    If $X \sim N(0, \sigma^2)$, then $E(\cos tX) = e^{\frac{-t^2\sigma^2}{2}}$.
    So $E[\cos X] = e^{-2}$

14. Let X be a normal random variable with mean 0 and variance 2. Then

    A. $E[\cos 2X] = e^{-4}$

    B. $E[\cos 2X] = 1$

C. $E[\cos 2X] = 0$

D. $E[\cos 2X] = e^{-2}$

Solution:
If $X \sim N(0, \sigma^2)$, then $E(\cos tX) = e^{\frac{-t^2\sigma^2}{2}}$.
So $E[\cos 2X] = e^{-4}$

15. If $f(x) = c\,exp^{[-(x^2-6x+9)/32]}$ represents pdf of a normal distribution, then the value of c, μ and σ are respectively

   (a) $(1/4, 2, 4)$

   (b) $(\frac{1}{4\sqrt{2\pi}}, 3, 4)$

   (c) $(\frac{4}{\sqrt{2\pi}}, 3, 2)$

   (d) $(\frac{1}{4\sqrt{2\pi}}, 4, 3)$

   Solution:
   If X follows normal with mean μ and variance σ^2, then its pdf is given by
   $f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp^{\frac{-(x-\mu)^2}{2\sigma^2}}$.
   By comparing with the given $f(x)$, we get $\mu = 3$ and $2\sigma^2 = 32$ implies $\sigma = 4$ and $c = \frac{1}{4\sqrt{2\pi}}$.

16. The life time (in hours) of a bulb follows exponential distribution with meanθ which is unknown. The probability that the bulb survive 12 hours is 0.32. Hence the probability that a bulb which is used for 12 hours will survive additional 12 hours is

   (a) 0.64

   (b) 0.16

   (c) 0.32

   (d) 0.42

   Solution:
   Here X follows exponential with $P[X \geq 12] = 0.32$
   We have to find $P[X \geq 12 + 12 | X \geq 12]$.
   Since X satisfies lack of memory property, we get $P[X \geq 12 + 12 | X \geq 12] = P[X \geq 12] = 0.32$

17. Suppose the amount of time one spends in a bank is exponentially distributed with mean 10 minutes. Then the probability that a customer will spend more than 15 minutes in the bank given that he has already spent 10 minutes in the bank is

   (a) $e^{-3/2}$

   (b) $e^{-1/2}$

   (c) $\frac{1}{10}e^{-3/2}$

   (d) $\frac{1}{10}e^{-1/2}$

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Solution:

We have to find $P[X \geq 15|X \geq 10]$.

Since X satisfies lack of memory property, we get $P[X \geq 10+5|X \geq 10] = P[X \geq 5]$

But $P[X \geq 5] = e^{5-\lambda}$, where $\lambda = 1/mean = 1/10$.

Hence $P[X \geq 15|X \geq 10] = P[X \geq 5] = e^{-1/2}$

18. The lengths of leaves of a tree follows normal distribution with mean length 5cm and standard deviation of 0.5 cm. The probability that in a random sample of 2 leaves selected from this tree, both have length exceeding 5.98 cm is

    (a) zero

    (b) 0.05

    (c) 0.025

    (d) 0.000625

    Solution:

    If X_1 and X_2 represents two leaves which are selected, then we want to find $P[X_1 > 5.98, X_2 > 5.98]$.

    As X_1 and X_2 are independent (as lengths of leaves are independent), we get

$$\begin{aligned} P[X_1 > 5.98, X_2 > 5.98] &= P[X_1 > 5.98]P[X_2 > 5.98] \\ &= P[Z_1 > 1.96]P[Z_2 > 1.96] \\ &= (1 - \Phi(1.96))^2 = (0.025)^2 = 0.000625 \end{aligned}$$

19. The number of accidents per week in a city has Poisson distribution with mean 3. What is the probability of exactly 2 accidents in 2 weeks?

    (a) $2e^{-3}$

    (b) $2e^{-6}$

    (c) e^{-6}

    (d) $18e^{-6}$

    Solution:

    Suppose X and Y represents number of accidents in week 1 and 2 , respectively.

    Then X and Y are independent Poisson random variables with $\lambda = 3$.

    And the probability of exactly 2 accidents in 2 weeks is

$$\begin{aligned} P[X=2, Y=0] + P[X=0, Y=2] + P[X=1, Y=1] &= P[X=2]P[Y=0] + P[X=0]P[Y=2] + P[X=1]P[Y=1] \\ &= 9e^{-6} + 9e^{-6} \\ &= 18e^{-6} \end{aligned}$$