1 Discrete random variables

Definition 1.1 (**Discrete Random Variable**). A r.v. $X : \Omega \longrightarrow R$ is said to be discrete if the set $\{X(w)|w \in \Omega\}$ is countable i.e. if X takes countably many values.

Example 1.2. 1. Consider an experiment of tossing a coin twice

Then $\Omega = \{HH, HT, TH, TT\}$

Define $X : \Omega \longrightarrow R$ by $X(\Omega) =$ no. of heads occurred in the outcome w

$\therefore X(w) \in 0, 1, 2 \Rightarrow X$ is a discrete r.v.

2. $X(w) =$ The sum of faces of the two dice $X((i, j)) = i + j = \{2, 3, ...12\} \Rightarrow X$ is a discrete r.v.

Definition 1.3 (Probability mass function **(p.m.f.)**). A function $p : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be probability mass function if it satisfies,

- $p(x) \geq 0$
- $\sum_x p(x) = 1$

Theorem 1.4. *Let X be a discrete random variable on $(\Omega, \mathcal{A}, P)$, with its range set $\{x_1, x_2, ...\}$. Then the function $p : \mathbb{R} \to \mathbb{R}$ defined by*

$$p(x) = \begin{cases} P[X = x_i], & x = x_i \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

is a probability mass function.

Proof. If $x = x_i$ for some i, then $p(x) = P[X = x_i] \geq 0$
If $x \neq x_i$ for any i, then $p(x) = 0$. Thus $p(x) \geq 0$.

$$\begin{aligned} \text{And } \sum_x p(x) &= \sum_{x_i} P[X = x_i] \\ &= P\Big(\bigcup_{x_i} [X = x_i]\Big) \\ &= P(\Omega) = 1 \end{aligned}$$

Therefore, $p(x)$ is a p.m.f. □

Definition 1.5 (Probability mass function **(p.m.f.)** of a r.v.). Let X be a discrete random variable, with its range set $\{x_1, x_2, ...\}$. Then the function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = \begin{cases} P[X = x_i], & x = x_i A \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

is called the probability mass function (p.m.f.) of the random variable X.

Remark 1.6. The distribution function of a r.v. X is

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \sum_{x_i \leq x} P(X = x_i) \\ &= \sum_{a \leq x} p(x_i). \end{aligned}$$

Exercise:

1. Verify whether $p(x) = 2(\frac{1}{3})^x \ x = 1, 2, \ldots$ is a p.m.f.

2. $p(x) = (\frac{1}{2})^x, \quad x = 0, 1, 2, ...$ is not a p.m.f.

3. $p(x) = \frac{e^{-1}}{|x|!}, \quad x = 0, \pm 1, \pm 2, ...$ is not a p.m.f.

4. Find k s.t. $p(x)$ is a p.m.f.

   (a) $p(x) = \frac{k}{x^2}, \quad x = 1, 2, 3, \ldots$

   (b) Find k s.t. $p(x) = \frac{k}{x(x+1)}, \quad x = 1, 2, \ldots$

   (c) $p(x) = \frac{ke^{-1}}{x!}$

   (d) $p(x) = \frac{k}{5^x}(x+1), \quad x = 0, 1, 2, \ldots$

Problem: The probability distribution of a r.v. X is given by

x	1	2	3	4
$p(x)$	k	$11k$	$3k$	$5k$

.

Find $P(x \leq 2)$.

Solution: Here X is discrete r.v.

$$\sum p(X = x_i) = 1$$
$$\Rightarrow 20k = 1$$
$$\Rightarrow k = \frac{1}{20}$$
$$\therefore P(X \leq 2) = P(X = 1) + P(X = 2)$$
$$= 12k$$
$$= \frac{12}{20}$$
$$= \frac{3}{5}$$

Example 1.7. Suppose X has a p.m.f. given by $p(1) = \frac{1}{2}$, $p(2) = \frac{1}{3}$, $p(3) = \frac{1}{6}$. Find the distribution function F of X.

Solution: The distribution function $F(x) = P(X \leq x)$

If $x < 1$, then $P(X \leq x) = P(\emptyset) = 0$

If $1 \leq x < 2$, then $P(X \leq x) = P(X = 1) = \frac{1}{2}$

If $2 \leq x < 3$, then $P(X \leq x) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

If $x \geq 3$, then $P(X \leq x) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$

Hence, $F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{5}{6}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$

2 Continuous random variables

Definition 2.1 (**Continuous Random Variable**). A r.v. X is said to be continuous if its set of possible values (range set of X) is an interval.

Definition 2.2 (**Probability density function (p.d.f.)**). Let X be a continuous r.v. then a function $f : \mathbb{R} \to \mathbb{R}$ is said to be p.d.f. if

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

having the property that for any set B of real numbers
$P[X \in B] = \int_B f(x)dx$.

Remark 2.3.
1. If $B = [a, b]$, $P[a \leq x \leq b] = \int_a^b f(x)dx$
   In particular if $a = b$, then $P[X = a] = \int_a^a f(x)dx = 0$.
   Therefore, if X is continuous, then probability at a point is zero.
2. The distribution function

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(X \in (-\infty, x]) \\ &= \int_{-\infty}^{x} f(t)dt \end{aligned}$$

Example 2.4. Verify weather the following function $f(x)$ functions $f(x)$ is p.d.f

1. $f(x) = \frac{1}{\theta} e^{\frac{-x}{\theta}} \quad \theta > 0, x \geq 0$
   (1)Clearly $\quad f(X) \geq 0$,
   (2)$\int_0^{\infty} f(x)dx = \frac{1}{\theta} \int_0^{\infty} e^{\frac{-x}{\theta}} dx = \frac{1}{\theta} [-\theta \frac{e^{\frac{-x}{\theta}}}{]}_0^{\infty} = 1$
2. $f(x) = \frac{1}{b-a} \quad a < x < b$
   (1)$f(x) \geq 0$
   (2)$\int_a^b f(x)dx = \frac{1}{b-a} \int_a^b 1dx = (\frac{1}{b-a})(b - a) = 1$

Example 2.5. Verify whether the function $f(x) = \frac{1}{x^2} \quad 1 \leq x < \infty$ is a p.d.f. or not. If it is a p.d.f., find $P(A), P(B), P(A \cap B), P(A \cup B)$, where $A = \{x|1 \leq x \leq 2\}$ $B = \{X|1 \leq X \leq 4\}$. Are A and B independent?
Solution Clearly, $f(x) \geq 0$, Now

$$\begin{aligned} \int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{1}{x^2} dx \\ &= [\frac{-1}{x}]_1^{\infty} \\ &= [0 - (-1)] = 1. \end{aligned}$$

Hence f is a p.d.f.

$$Then\, P(A) = P(1 \leq x \leq 2) == \int_1^2 f(x)dx$$

$$= \int_1^2 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^2$$
$$= [\frac{-1}{2} + 1] = \frac{1}{2}$$

Similarly,

$$P(B) = P[1 \leq X \leq 4] = = \int_1^4 \frac{1}{x^2} dx$$
$$= [\frac{-1}{x}]_1^4$$
$$= [\frac{-1}{4}] = \frac{3}{4}$$

Since $A \cup B = B$, $P(A \cup B) = P(B) = \frac{3}{4}$
And as $A \cap B = A$, $P(A \cap B) = P(A) = \frac{1}{2}$
Since $P(A \cap B) = \frac{1}{2} \neq \frac{3}{8} = P(A)P(B)$, A and B are not independent.

Exercise Find k s.t $f(x)$ is a p.d.f

- $f(x) = ke^{-\alpha x} x^{\lambda - 1} \quad x, \alpha, \lambda > 0$
- $f(x) = \frac{k}{1+x^2} \quad -\infty < x < \infty$
- $f(x) = ke^{-\lambda |x|} \quad -\infty < x < \infty$.

Definition 2.6 (**Joint Distribution Function**)**.** Let X and Y be two random variables. Then the joint (cumulative) distribution function of X and Y is the function on $\mathbb{R}^2$ defined by,
$F(x, y) = P(X \leq x, Y \leq y) \quad (x, y) \in \mathbb{R}^2$.

Remark 2.7. Suppose $F(x, y)$ is the joint distribution function of X and Y. Then we get the distribution function F_X of X as:

$$F_X(x) = P(X \leq x)$$
$$= P(X \leq x, Y < \infty)$$
$$= \lim_{y \to \infty} P(X \leq x, Y \leq y)$$
$$= \lim_{y \to \infty} F(x, y)$$
$$= F(x, \infty)$$

The distribution function F_X is called the marginal distribution of X.

Definition 2.8 (**Marginal Distribution Functions**)**.** Suppose $F(x, y)$ is the joint distribution function of X and Y. Then

1. The marginal distribution function F_X of X is defined as
   $F_X(x) = \lim_{y \to \infty} F(x, y) = F(x, \infty)$
2. The marginal distribution function F_Y of Y is defined as
   $F_Y(y) = \lim_{x \to \infty} F(x, y) = F(\infty, y)$

Definition 2.9 (Joint p.m.f of X). Let X and Y be two discrete random variables. Then the joint probability mass function $p(x, y)$ of X and Y is the function on $\mathbb{R}^2$ defined by,
$p(x, y) = P(X = x, Y = y) \quad (x, y) \in \mathbb{R}^2$.

Remark 2.10. Suppose $p(x, y)$ is the joint probability mass function of X and Y.
Since Y must take on some value y_j, we get that
$\{X = x\} = \bigcup_j \{X = x, Y = y_j\}$.
Hence the pmf p_X of X is:

$$\begin{aligned} P_X(x) &= P(X = x) \\ &= P\Big(\bigcup_j \{X = x, Y = y_j\}\Big) \\ &= \sum_j P(X = x, Y = y_j) \\ &= \sum_j p(x, y_j) \end{aligned}$$

The pmf p_X is called the marginal probability mass function of X.

Definition 2.11 (Marginal p.m.f.). Suppose $p(x, y)$ is the joint probability mass function of X and Y. Then

1. The marginal probability mass function p_X of X is defined as
   $p_X(x) = P(X = x) = \sum_j p(x, y_j) \quad x \in \mathbb{R}$
2. The marginal probability mass function p_X of X is defined as
   $p_Y(y) = P(Y = y) = \sum_i p(x_i, y) \quad y \in \mathbb{R}$

Definition 2.12 (Joint p.d.f. of X). Let X and Y be two continuous random variables. Then the joint probability density mass function $f(x, y)$ of X and Y is the function on $\mathbb{R}^2$ such that for any region C in $\mathbb{R}^2$, we have
$P((x, y) \in C) = \iint_{(x,y)\in C} f(x, y)dxdy$
In particular, if A and B are subsets of $\mathbb{R}$ then,
$P(x \in A, y \in B) = \int_A \int_B F(x, y)dydx$.

Remark 2.13.

1. The distribution function of X and Y is
   $F(x, y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)dydx$
   If partial derivatives of $F(x, y)$ exists, then we get
   $f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y)$
2. The pdf of X can be obtained as

$$\begin{aligned} P(X \in A) &= P(X \in A, Y \in (-\infty, \infty)) \\ &= \int_A \int_{-\infty}^{\infty} f(x, y)dydx \\ &= \int_A f_X(x)dx \end{aligned}$$

   Where $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is called the marginal density function of X.

Definition 2.14 (Marginal p.d.f.). Suppose $f(x, y)$ is the joint pdf of X and Y. Then

1. $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is called the marginal density function of X.
2. $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ is called the marginal density function of Y.

Example 2.15. Joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & x < 0 < \infty \\ 0, & otherwise \end{cases}$$

Compute $P(X > 1, Y > 1)$, $P(X < Y)$ and $P(X < a)$

Proof.

$$\begin{aligned} P(X > 1, Y < 1) &= \int_1^{\infty}\int_0^1 F(x, y)dydx \\ &= \int_1^{\infty}\int_0^1 2e^{-x}e^{-2y}dydx \\ &= \int_1^{\infty} [2e^{-x}e^{-2y}]_0^1 \\ &= \int_1^{\infty} e^{-x}[\frac{e^{-2}}{-2} - \frac{1}{-2}]dx \\ &= \int_1^{\infty} e^{-x}(e^{-2} - 1)dx \\ &= (e^{-2} - 1)[\frac{e^{-x}}{-1}]_1\infty \\ &= (e^{-2} - 1)(0 - e^{-1}) \\ &= \frac{-1}{e^3} + \frac{1}{e} = \frac{1}{e}(1 - \frac{1}{e^2}) \\ P(X < Y) &= \int_0^{\infty}\int_0^y F(x, y)dydx \\ &= \int_0^{\infty} [\frac{2e^{-x}e^{-2y}}{-1}]_0^y dy \\ &= \int_0^{\infty} -2e^{-2y}(e^{-y} - e^0)dy \\ &= \int_0^{\infty} (-2e^{-3y} + 2e^{-2y})dy \\ &= [\frac{-2}{e}^{-3y} 3 + e^{-2y}]_0^{\infty} \\ &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} P(X < a) &= \int_0^{\infty}\int_0^a F(x, y)dydx \\ &= \int_0^{\infty}\int_0^a 2e^{-x}e^{-2y}dxdy \\ &= \int_0^{\infty} [\frac{2e^{-x}}{-1}]_0^a e^{-2y}dy \\ &= \int_0^{\infty} -2[e^{-a} - 1]e^{-2y}dy \\ &= -e^{-a} + 1 = 1 - \frac{1}{e^a} \end{aligned}$$

□

Definition 2.16 (Independent Random Variables). Two random variables X and Y are said to be independent if for any two set of real numbers A and B $P(X \in A, Y \in B) = P(X \in A).P(Y \in B)$.

Remark 2.17.
1. X and Y are independent if $F(a, b) = F_X(a)F_Y(b)$.
2. X and Y are independent if for all $a, b \in \mathbb{R}$, $P(X \le a, Y \le b) = P(X \le a)p(Y \le B)$.
3. Two discrete r.vs X and Y are independent if $P(x, y) = P_X(x)P_Y(y)$
4. Two continuous r.v's X and Y are independent if $f(x, y) = f_X(x)f_Y(y) \quad \forall x, y$

Example 2.18. Suppose that X and Y are independent r.v having the common density function f_x given by
$f(x) = \begin{cases} e^{-x}, & \text{x} > 0 \\ 0, & \text{otherwise.} \end{cases}$
find the density function of the r.v X/Y

Proof. First we find the distribution function of X/Y

$$\begin{aligned} F_{X/Y}(a) &= P(X/Y \le a) \\ &= \iint\limits_{\frac{x}{y} \le a} f(x)f(y)dxdy \\ &= \iint\limits_{x \le ay} e^{-x}e^{-y}dxdy \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty}\int_0^{ay} e^{-y}e^{-x}dxdy \\ &= \int_0^{\infty} e^{-y}(1 - e^{-ay})dy \\ &= \left[-e^{-y} + \frac{e^{-(a+1)y}}{a+1}\right]_0^{\infty} \\ &= 1 - \frac{1}{a+1} \end{aligned}$$

Now by differentiating $F_{X|Y}(a)$, we get $f_{\frac{X}{Y}}(a) = (1 + a)^{-2} \quad 0 < a < \infty$ □

Definition 2.19 (Conditional distribution).
1. If X and Y are discrete random variables, then the conditional p.m.f of X given that $Y = y$ is defined as

$$\begin{aligned} p_{X|Y}(x|y) &= P(X = x|Y = y) \\ &= \frac{P(X = x, Y = y)}{P(Y = y)} \\ &= \frac{p(x, y)}{p_Y(y)} \end{aligned}$$

2. If X and Y are continuous random variables, then the conditional p.d.f of X given that $Y = y$ is defined as $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

Example 2.20. Suppose that $p(x,y)$ is the joint p.m.f. of (x,y) given by $p(0,0) = 0.4, p(0,1) = 0.2, p(1,0) = 0.1, p(1,1) = 0.3$. calculate the conditional p.m.f. of X given that $Y = 1$.

Proof. $p_{X|Y}(x|1) = P(X = x|Y = 1) = \frac{P(X=x,Y=1)}{P(Y=1)} = \frac{p(x,1)}{p_Y(1)}$
We have $p_y(1) = p(0,1) + p(1,1) = 0.5$
$\Rightarrow P(X = 0|Y = 1) = P_{X|Y}(0|1) = \frac{p(0,1)}{p_y(1)} = \frac{0.2}{0.5} = \frac{2}{5}$,
Similarly $p(X = 1|Y = 1) = P_{X|Y}(1|1) = \frac{p(1,1)}{p_y(1)} = \frac{0.3}{0.5} = \frac{3}{5}$. □

Example 2.21. The joint p.d.f. of x and y is given by
$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & \text{if } 0 < x < 1 \\ 0 & \text{if } otherwise \end{cases}$ Compute conditional density function of X given by $Y = y$.

Proof. $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ Now,

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x,y)dx \\ &= \int_0^1 \frac{12}{5}x(2-x-y)dx \\ &= \frac{12}{5}\int_0^1 (2x - x^2 - xy)dx \\ &= \frac{12}{5}(x_2 - \frac{x^3}{3} - \frac{x_2 y}{2} = \frac{12}{5}(1 - \frac{1}{3} - \frac{y}{2}) \\ &= \frac{12}{5}(\frac{2}{3} - \frac{y}{2}) = \frac{12}{5}(\frac{4-3y}{6}) \\ &= \frac{\frac{12}{5}x(2-x-y)}{\frac{12}{5}(\frac{4-3y}{6})} \\ f_{X|Y}(x|y) &= \frac{6x(2-x-y)}{4-3y}. \end{aligned}$$

□

3 Expectation

Let X be real valued random variable with distribution function F_x. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then $g(x)$ is a r.v. and the expectation of the r.v. $g(x)$ denoted by $E(g(X))$ is defined as
$E(g(X)) = \begin{cases} \sum_{x\in X} g(x)p(X = x) & \text{if X is discrete r.v. with p.m.f. p(x)} \\ \int_x g(x)F(x)dx & \text{if X is continuous function} \end{cases}$
In particular by considering $g(x) = x$, we get the expectation of X as
$E(X) = \begin{cases} \sum_x xp(X = x) & \text{if X is discrete r.v. with p.m.f. p(x)} \\ \int_x xF(x)dx & \text{if X is continuous function} \end{cases}$

Example 3.1.
1. If the p.m.f. of X is given by $p(0) = \frac{1}{2} = p(1)$, then
   $E(X) = 0p(0) + 1p(1) = 0 + \frac{1}{2} = \frac{1}{2}$
2. Find $E(X)$ where X is the outcome when we role a fair die.
   Solution: $X(\omega) \in \{1,2,\ldots,6\} \Rightarrow p(i) = \frac{1}{6} \quad \forall i = 1,2,\ldots,6$.
   Then $E(X) = \sum xp(x_i) = \frac{1}{6}[1+2+3+4+5+6 = \frac{1}{6}(21) = \frac{7}{2}$

3. For a measurable set A, consider the random variable I_A.

   Since I_A takes two values 0 on A^c and 1 on A, we get $E(I_A) = 0P(A^c) + aP(A) = P(A)$.

   Thus expectation of the indicator function of A is the probability of A.

Remark 3.2. The expected value of X is not a value that X could possible assume, it should be average value of X in a large number of repetitions of the experiment. The expected value of X is a weighted average of the possible values that X can take on each value being weighted by the probability that X assume it.

Example 3.3. 1. Suppose that you are expecting a message some time past 5 pm. From experience you know that X, the number of hours after 5 pm until the message arrive is a random variable with the following p.d.f. $f(x) = \begin{cases} \frac{1}{1.5} & \text{if } 0 < x < 1.5 \\ 0 & \text{otherwise} \end{cases}$
Then the expected amount of time past 5 pm until the message arrive is given by

$$\begin{aligned} E(X) &= \int_0^{1.5} xf(x)dx \\ &= \frac{1}{1.5}[\frac{x^2}{2}]_0^{1.5} \\ &= \frac{1.5 \times 1.5}{2 \times 1.5} \\ &= \frac{1.5}{2} \\ &= 0.75 \end{aligned}$$

Hence on average you would have to wait (3|4) three-fourths of an hour.

2. Suppose X has the following p.m.f. $P(0) = 0.5, P(1) = 0.5, P(2) = 0.3$ calculate $E(X^2)$

   Solution:

$$\begin{aligned} E(X^2) &= \sum_{x=0}^{2} x^2 p(x) \\ &= 1.p(1) + 2^2 p(2) \\ &= 1 \times 0.5 + 4 \times 0.3 \\ &= 0.5 + 1.2 \\ &= 1.7 \end{aligned}$$

3. The time in hours, it takes to locate and repair an electrical break down in a certain factory is a r.v. call it X, whose density function is given by,
   $f_X(x) = \begin{cases} 1 & 0 < x < 1; \\ 0 & otherwise; \end{cases}$
   If the cost involved in a break down of duration x is x^3, what is expected cast of such break down?

   Solution: Let $Y = x^3$ denote the cost.

   For any $0 \le a \le 1$, we find the distribution function

$$\begin{aligned} F_y(a) &= P(y \le a) \\ &= P(X^3 \le a) \end{aligned}$$

$$
\begin{aligned}
&= P(X \leq a^{1/3}) \\
&= \int_0^{a^1/3} f(x)dx \\
&= \int_0^{a^1/3} 1dx \\
&= a^1/3
\end{aligned}
$$

By differentiation $F_y(a)$, we get the density function,

$$
\begin{aligned}
f_Y(a) &= \frac{1}{3}a^{\frac{-2}{3}} \qquad 0 \leq a \leq 1 \\
Hence,\ E(Y) &= \int_0^1 y f_y(y)dy \\
&= \int_0^1 y\frac{1}{3}y^{\frac{-2}{3}}dy \\
&= \frac{1}{3}[\frac{y^{\frac{4}{3}}}{4/3}]\lim_0^1 \\
&= \frac{1}{4}.
\end{aligned}
$$

Theorem 3.4. $E(ax + b) = aE(x) + b, \qquad \forall a, b \in R.$
In particular, $E(b) = b$, expectation of constant is itself and $E(aX) = aE(X)$.

Proof. Take $g(x) = ax + b$. Then $g(x)$ is continuous and hence $g(aX + b)$ is a random variable.

- Suppose X is discrete,

$$
\begin{aligned}
E(aX + b) &= \sum (ax + b)P(X = x) \\
&= \sum_x axP(X = x) + \sum bP(X = x) \\
&= a\sum xP(X = x) + b\sum P(X = x) \\
&= aE(X) + b
\end{aligned}
$$

- Suppose X is continuous

$$
\begin{aligned}
E(aX + b) &= \int_x (ax + b)f(x)dx \\
&= \int_x axf(x)dx + \int_x bF(x)dx \\
&= a\int_x f(x)dx + b\int_x f(x)dx \\
&= aE(X) + b.
\end{aligned}
$$

□

Definition 3.5. (**Moments of a r.v.X:**)
Expectation $E(X^n)$ of the random variable X^n is called the n^{th} moments of X. For $n = 1$, $E(x)$ is called the first moment of X, also called as the mean of X, denoted by $\mu = E(x)$.

Definition 3.6. (**Moment Generating Function**)

The function $M_X(t) = E(e^{tX}), \quad t \in \mathbb{R}$, is called the moment generating function of X.

Remark 3.7. The moment generating function M_X generates all moment of X as follows

The n^{th} derivative of M_X is $M_X^{(n)}(t) = E(X^n e^{tX})$

Hence the n^{th} moment of X is $E(X^n) = M_X^{(n)}(0)$

That justifies the name of the function.

Theorem 3.8. *If X and Y are independent random variables then,*

1. $M_{X+Y}(t) = M_X(t)M_Y(t)$.
2. $M_{aX+b}(t) = e^{tb}M_{aX}(t)$

Proof. $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX}e^{tY})$

If X and Y are independent then for any continuous f, the random variables $f(X)$ and $f(Y)$ are independent. Since, $f(X) = e^{tx}$ is continuous, $f(X) = e^{tx}$ and $f(Y) = e^{ty}$ are independent random variables. Hence

$$\begin{aligned} M_{X+Y}(t) &= E(e^{tX}e^{tY}) \\ &= E(e^{tX})E(e^{tY}) \\ &= M_X(t)M_Y(t) \\ M_{aX+b}(t) &= E(e^{t(aX+b)}) \\ &= E(e^{taX}e^{tb}) \\ &= e^{tb}E(e^{taX}) \\ &= e^{tb}M_{aX}(t) \end{aligned}$$

□

Definition 3.9. (**Expectation of sum of r.v.**)

If X and Y are two random variables and $g(x, y)$ is a continuous function in two variables then,

$$E(g(x,y)) = \begin{cases} \sum_y \sum_x g(x,y)P(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_y \int_x g(x,y)f(x,y)dxdy & \text{if } X \text{ and } Y \text{ are continous} \end{cases}$$

Theorem 3.10. $E(aX + bY) = aE(X) + bE(Y)$.

Proof. 1. Discrete Case:

$$\begin{aligned} E(aX + bY) &= \sum_y \sum_x (ax + by)P(x,y) \\ &= \sum_y \sum_x axP(x,y) + byP(x,y) \\ &= a\sum_y \sum_x xP(x,y) + b\sum_y \sum_x yP(x,y) \end{aligned}$$

If we take $g(x, y) = x$, then

$$E(X) = E(g(x,y)) = \sum_y \sum_x g(x,y)P(x,y)$$

$$= \sum_y \sum_x xP(x,y)$$

Similarly, for $g(x,y) = y$ we get

$E(Y) = E(g(x,y)) = \sum_y \sum_x yP(x,y)$ Hence $E(aX+bY) = aE(X) + bE(Y)$.

2. Continuous Case:

$$\begin{aligned} E(aX+bY) &= \int_y \int_x (ax+by)f(x,y)dxdy \\ &= a\int_y \int_x xf(x,y)dxdy + b\int_y \int_x yf(x,y)dxdy \end{aligned}$$

If we take $g(x,y) = x$, then

$$\begin{aligned} E(X) + E(aX+bY) &= \int_y \int_x g(x,y)f(x,y)dxdy \\ &= \int_y \int_x xf(x,y)dxdy \end{aligned}$$

Similarly, for $g(x,y) = y$ we have

$E(Y) = \int_y \int_x yf(x,y)dxdy$ Thus $E(aX+bY) = aE(X) + bE(Y)$

□

Corollary 3.11. $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$

Proof. The result is true for $n = 2$ (by above result). Assume that result is true for $n = k$ i.e.

$$E(\sum_{i=1}^k X_i) = \sum_{i=1}^k E(X_i)$$

Then,

$$\begin{aligned} E(\sum_{i=1}^{k+1} X_i) &= E(\sum_{i=1}^{k} X_i + X_{k+1}) \\ &= E(\sum_{i=1}^{k} X_i) + E(X_{k+1}) \\ &= \sum_{i=1}^{k} E(X_i) + E(X_{k+1}) \\ &= \sum_{i=1}^{k+1} E(X_i) \\ \Rightarrow E(\sum_{i=1}^{n} X_i) &= \sum_{i=1}^{n} E(X_i). \end{aligned}$$

□

Definition 3.12 (Variance of random variable). If X is random variable with mean μ then the variance of X, denoted by $Var(X)$ is defined by $Var(X) = E((X-\mu)^2) = E((X-E(X))^2)$.

Theorem 3.13. *Let X be a random variable. Then*

1. $Var(X) = E(X^2) - E^2(X)$.

2. $Var(aX+b) = a^2Var(X)$.

   In particular, $Var(a) = 0$, *variance of a constant is zero and* $Var(aX) = a^2Vae(X)$.

Proof.

$$\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= E(X^2 + \mu^2 - 2\mu X) \\
&= E(X^2) + \mu^2 - 2\mu X \\
&= E(X^2) + \mu^2 - 2\mu^2 \\
&= E(X^2) - E^2(X) \\
Var(aX+b) &= E((aX+B)^2) - E^2(aX+B) \\
&= E(a^2X^2 + 2abX + b^2) - [aE(X)+b]^2 \\
&= a^2E(X^2) + b^2 + 2abE(X) - a^2E^2(X) - 2abE(X) - b^2 \\
&= a^2(E(X^2) - E^2(X)) \\
&= a^2Var(X).
\end{aligned}$$

□

Definition 3.14. For a random variable X, the square root of X is called the standard deviation of X. i.e. $\sigma_X = \sqrt{Var(X)} = \sqrt{[E(X^2) - E^2(X)]}$.

Definition 3.15. **(Co-Variance)**

The covariance of two random variable X and Y denoted by $Cov(X,Y)$, is defined as

$Cov(X,Y) = E((X-\mu_X)(Y-\mu_Y))$,

where $\mu_X = E(X)$ and $\mu_Y = E(Y)$.

Remark 3.16. 1. If $X = Y$ then $Cov(X,X) = E((X-\mu_X)(X-\mu_X)) = Var(X)$

2. $Cov(Y,X) = Cov(X,Y)$

Theorem 3.17. *1.* $Cov(X,Y) = E(XY) - E(X)E(Y)$

2. $Cov(aX,Y) = aCov(X,Y) = Cov(x,aY)$

3. $Cov(X_1+X_2,Y) = Cov(X_1,Y) + Cov(X_2,Y)$

4. In general, $Cov(\sum_{i=1}^{n} X_i, Y) = \sum_{i=1}^{n} Cov(X,Y_i)$

5. $cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_j \sum_i cov(X_i,Y_j)$

Proof.

$$\begin{aligned}
1.\, Cov(X,Y) &= E(X-\mu_X)(Y-\mu_Y) \\
&= E(XY - \mu_Y X - \mu_X Y + \mu_X\mu_X\mu_Y) \\
&= E(XY) - \mu_Y E(X) - \mu_X E(y) + \mu_X\mu_Y
\end{aligned}$$

$$
\begin{aligned}
&= E(XY) - \mu_Y\mu_X - \mu_X\mu_Y + \mu_X\mu_Y \\
2.\, Cov(aX, Y) &= E(aXY) - E(aX)E(Y) \\
&= aE(XY) - aE(X)E(Y) \\
&= a[E(XY) - E(X)E(Y)] \\
&= aCov(X, Y) \\
3.\, Cov(X_1 + X_2, Y) &= E((X_1 + X_2)Y) - E(X_1 + X_2)E(Y) \\
&= E(X_1Y + X_2Y) - (E(X_1) + E(X_2))E(Y) \\
&= E(X_1Y) + E(X_2Y) - E(X_1)E(Y) - E(X_2)E(Y) \\
&= E(X_1Y) - E(X_1)E(Y) + E(X_2Y) - E(X_2)E(Y) \\
&= Cov(X_1, Y) + Cov(X_2, Y) \\
\text{By induction}, Cov(\sum_{i=1}^{n} X_i, Y) &= \sum_{i=1}^{n} Cov(X_i, Y) \\
\text{Then}, Cov(X, \sum_{i=1}^{n} Y_i) &= Cov(\sum_{i=1}^{n} Y_i, X) \\
&= \sum_{i=1}^{n} Cov(Y_i, X) \\
&= \sum_{i=1}^{n} Cov(X, Y_i) \\
\text{Hence}, cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) &= \sum_{i=1}^{n} Cov(X_i, \sum_{j=1}^{m} Y_j) \\
&= \sum_{j} \sum_{i} cov(X_i, Y_j)
\end{aligned}
$$

□

Corollary 3.18. $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} Cov(X_i, X_j)$.

Proof.

$$
\begin{aligned}
Var(\sum_{i=1}^{n} X_i) &= Cov(\sum_{i=1}^{n} X_i, \quad \sum_{j=1}^{n} X_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Cov(X_i, X_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} Cov(X_i, X_j)
\end{aligned}
$$

□

Theorem 3.19. *If X and Y are independent then $Cov(X, Y) = 0$.*

Proof. If X and Y are independent, we show that $E(XY) = E(X)E(Y)$

1. Discrete case:

$$\begin{aligned} E(XY) &= \sum_x \sum_y xyP(x,y) \\ &= \sum_x \sum_y xyP(X=x, Y=y) \\ &= \sum_x \sum_y xyP(X=x)P(Y=y) \\ &= \sum_x xP(X=x) \sum_y yP(Y=y) \\ &= E(X)E(Y) \end{aligned}$$

2. Continuous case:

$$\begin{aligned} E(XY) &= \int_x \int_y xyf(x,y)dydx \\ &= \int_x \int_y xyf_X(x)f_Y(y)dydx \\ &= \int_x xf_X(x)dx \int_y yf_Y(y)dy \\ &= E(X)E(Y) \end{aligned}$$

$$\begin{aligned} \therefore Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(X)E(Y) - E(X)E(Y) \\ &= 0 \end{aligned}$$

□

Corollary 3.20. *If $X_1, X_2, \ldots, X_n$ are independent random variable, then*

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

Proof. Since X_i and $Y_j \quad i \neq j$ are independent, $Cov(X_i, X_j = 0 \, \forall i \neq j$.

$$\begin{aligned} \therefore Var(\sum_{i=1}^{n} X_i) &= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} Cov(X_i, X_j) \\ &= \sum_{i=1}^{n} Var(X_i). \end{aligned}$$

□

Definition 3.21 (Correlation:). Correlation between two random variables X, Y denotes by $Corr(X,Y)$ and is defined as,

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Remark 3.22. $-1 \leq Corr(X,Y) \leq 1$.

Theorem 3.23 (**Markov's inequality**). *If X is a random variable that takes only non-negative values.(i.e $x_i = X(\omega) \geq 0\ \forall\omega$), then for any $a > 0$,*
$P(X \geq a) \leq \frac{E(X)}{a}$.

Proof. Let $A = \{\omega_i | x_i = X(\omega_i) \geq a\}$

1. X is discrete:

$$
\begin{aligned}
E(X) &= \sum xP(X = x_i) \\
&= \sum_{x_i < a} x_i P(X = x_i) + \sum_{x_i \geq a} x_i P(X = x_i) \\
&\geq \sum_{x_i \geq a} x_i P(X = x_i) \\
&\geq \sum_{x_i \geq a} aP(X = x_i) \qquad (\because x_i \geq a) \\
&= a \sum_{x_i \geq a} P(X = x_i) \\
&= aP(A) \\
\therefore E(X) &\geq aP(A) \\
\Rightarrow P(A) &\leq \frac{E(X)}{a} \\
P(X \geq a) &\leq \frac{E(X)}{a}
\end{aligned}
$$

2. X is continuous.

$$
\begin{aligned}
E(X) &= \int_x xf(x)dx \\
&= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx \\
&\geq \int_a^\infty xf(x)dx \\
&\geq \int_a^\infty af(x)dx \\
&= a\int_a^\infty f(x)dx \\
&= aP(X \geq a) \\
P(X \geq a) &\leq \frac{E(X)}{a}.
\end{aligned}
$$

□

Theorem 3.24 (**Chebyshev's inequality**). *If X is a random variable with mean μ and variance σ^2 then for any value $k > 0, P[|X - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$.*

Proof. Since, $(x-\mu)^2$ is a non-negative random variable, by Markov's inequality applied for k^2,
We get,

$$P[(X-\mu)^2 \geq k^2] \leq \frac{E((X-\mu)^2)}{k^2}$$

But

$$[(X-\mu)^2 \geq k^2] = [|x-u| \geq k]$$

and $E((X-\mu)^2) = \sigma^2$,variance

$$\therefore P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

□

Example 3.25. Suppose that it is known that the number of items produced in factory during a week is a random variable with mean 50.

1. What can be said about the probability that this week's production exceed 75?
2. If the variance of a week's production is known to equal 25,then what can be said about the probability that this week's production between 40 and 60?

Solution: Let X be the no. of items that will produced in a week,

1. By,Markov's inequality,

$$P(x > 75) \leq \frac{E(X)}{75} = \frac{50}{75} = \frac{2}{3} = 0.6$$

2. $\sigma^2 = 25, \mu = 50$, we have to find $P(40 \leq X \leq 60)$ but

$$\begin{aligned} P(40 \leq X \leq 60) &= P(50-10 \leq X \leq 50+10) \\ &= P(|x-50| \geq 10) \end{aligned}$$

By Chebyshev's inequality,

$$P(|X-50| \geq 10) \leq \frac{\sigma^2}{10^2} = \frac{25}{100} = \frac{1}{4} = 0.25.$$

Theorem 3.26. *(**Jensen's inequality**) If X is a random variable and f is a convex function then $E(f(X) \geq f(E(X))$.*

Remark 3.27. 1. A function f is convex if and only if $-f$ is concave.

2. A function f is convex if $f''(x) > 0$

4 NET/SET questions

(1). The mean and variance of 25 is :

A. Mean=25, Variance=1

B. Mean=25, Variance=0

C. Mean=25, Variance=1/25

D. Mean=25, and Variance cannot be found

Solution: Mean of a constant random is the same constant and variance is zero.

(2). Let N be a random variable with $P(N=n) = \frac{1}{n(n+1)} \quad n = 1, 2, ...,$. Then $E[N]$ is:

A. 1/10

B. 3/10

C. 1

D. infinity

Solution: $E[N] = \sum_{n=1}^{\infty} nP(N=n) = \sum_{n=1}^{\infty} n\frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$

(3). Let X be a random variable with probability distribution: $P[X=k] = p_k, \quad k = 1, 2, ...$ Then:

A. $E[X] \leq \sum_{k=1}^{\infty} P[X \geq k]$

B. $E[X] = \sum_{k=1}^{\infty} P[X \geq k]$

C. $E[X] \geq \sum_{k=1}^{\infty} P[X \geq k]$

D. $E(X)$ does not exists.

Solution:

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} kP[X=k] \\
&= P[x=1] + 2P[X=2] + 3P[x=3] + 4P[X=4] + ... \\
&= (P[x=1] + P[X=2] + ...) + (P[X=2] + P[x=3] + ...) + (P[X=3] + P[x=4] + ...) \\
&= P[X \geq 1] + P[X \geq 2] + P[X \geq 3] + ... \\
&= \sum_{k=1}^{\infty} P[X \geq k]
\end{aligned}
$$

(4). A crop insurance company establishes the following loss table based upon previous claims:

percent loss	probability
0	0.90
25	0.05
50	0.02
100	?

If they write policy that pays a maximum of \$150/hectare, their expected loss in \$/hectare is approximately:

A. 5.2

B. 7.9

C. 4.5

D. 25.0

Solution: Sum of the probabilities equals to 1 gives the missing probability which is equal to 0.03 Hence the expected value is 5.25.

(5). The average length of stay in a hospital is useful for planning purpose. Suppose that the following is the distribution of the length of stay in a hospital after minor operation:

Days	probability
2	0.05
3	0.20
4	0.40
5	0.20
6	?

The average length of stay is:

A. 0.15

B. 2.25

C. 4.20

D. 5.50

Solution: Sum of the probabilities equals to 1 gives the missing probability which is equal to 0.15 Hence the expected value is 4.20.

(6). A random variable Y has the following probability distribution:

Y	$P(Y)$
-1	3C
0	2C
1	0.4
2	0.1

The value of the constant C is:

A. 0.05

B. 0.10

C. 0.15

D. 0.20

Solution: Given that $P[X=-1]=3C, P[X=0]=2C, P[X=1]=0.4, P[X=2]=0.1$. Sum of these probabilities is equal to 1, implies, $3C+2C+0.4+0.1=1 \Rightarrow C=0.10$

(7). The number of adults (X) living in homes on a randomly selected city block is described by the following probability distribution:

X	**Prob.**
1	0.25
2	0.50
3	0.15
≥ 4	?

What is the probability that 4 or more adults reside at randomly selected home?

A. 0.50

B. 0.25

C. 0.15

D. 0.10

Solution: Given that $P[X = 1] = 0.25, P[X = 2] = 0.50, P[X = 3] = 0.15$ and we have to find $P[X \geq 4]$. Sum of these probabilities is equal to 1, implies, $P[X \geq 4] = 1 - P[X = 1] - P[X = 2] - P[X = 3] = 0.10$

(8). An insurance company has estimated the following cost probabilities for the next year on a particular model of car:

Cost(Rs.)	**Prob.**
0	0.60
500	0.05
1000	0.13
2000	?

The expected cost to the insurance company is (approximately):

A. Rs. 155

B. Rs. 595

C. Rs.645

D. Rs.875

Solution: Given that $P[X = 0] = 0.60, P[X = 500] = 0.05, P[X = 1000] = 0.13$ and we have to find $P[X = 2000]$ and then $E(X)$. Sum of these probabilities is equal to 1, implies, $P[X = 2000] = 1 - P[X = 0] - P[X = 500] - P[X = 1000] = 0.22$. Hence the expected cost is $E(X) = \sum xP(X = x) = 0(0.60) + 500(0.05) + 1000(0.13) + 2000(0.22) = 595$.

(9). Let X be a discrete random variable with p.m.f is given by

x	-1	0	1	2
$P(x)$	1/4	1/4	1/4	1/4

Then

A. $E(X) = 0$ and $E(X^2) = 1$

B. $E(X) = 0$ and $E(X^2) = 1$

C. $E(X) = 0$ and $E(X^2) = 1$

D. $E(X) = 0$ and $E(X^2) = 1$

Solution:$E(X) = \sum xP(x) = 1/2$ $E(X^2) = \sum x^2 P(x) = 1.5$

(10). If $E(X)$, the expected value of a non degenerate r.v. X equals 3, which is always true ?

A. $E(X^2) = 9$

B. $E(X^2) = 6$

C. $E(X^2) < 9$

D. $E(X^2) \geq 9$

Solution: Since $V(X) \geq 0$, we get that $E(X^2) \geq E(X)^2 \Rightarrow E(X^2) \geq 9$.

(11). Let X be a r.v. with second moment, that is $E(X^2) = 9$. Then

A. $E(X) > 3$

B. $E(X) = 3$

C. $E(X) \leq 3$

D. nothing can be said about $E(X)$.

Solution: Since $V(X) \geq 0$, we get that $E(X)^2 \leq E(X^2) \Rightarrow E(X) \leq 3$.

(12). The sample mean of a sample of 100 observations is observed to be 10. The sample variance is given by 9. Then the standard error of the sample mean is given by

A. 0.3

B. 0.9

C. 0.09

D. 0.1

Solution: Here $n = 100$, $mean = \bar{x} = 10$ and $variance = \sigma^2 = 9$ Hence the standard error of mean=$\frac{\sigma}{\sqrt{n}} = \frac{3}{10} = 0.3$

(13). Let X be a discrete r.v. with $E(X) = 0$ and p.m.f with two missing probabilities, given below

x	-3	-1	0	1	2
$P[X = x]$	0.1	-	-	0.2	0.2

The the missing probabilities are

A. $P[X=-1]=0.2$ and $P[X=0]=0.3$

B. $P[X=-1]=0.3$ and $P[X=0]=0.2$

C. $P[X=-1]=0.3$ and $P[X=0]=0$

D. cannot be determined from the given information.

Solution: Let $P[X=-1]=a$ and $P[X=0]=b$
$\sum P[X=x]=1 \Rightarrow 0.5+a+b=1 \Rightarrow a+b=0.5$
And $E(X)=0 \Rightarrow -0.3-a+0.2+0.4=0 \Rightarrow a=0.3$ and so $b=0.2$

(14). If X is a r.v. with p.d.f.

$$f(x)=\begin{cases}1.4e^{-2x}+0.9e^{-3x} & x\geq 0\\ 0 & x<0\end{cases}$$

Then $E(X)$ equals:

A. 9/20

B. 5/6

C. 1

D. 23/10

Solution:

$$\begin{aligned}E(X)&=\int_0^\infty xf(x)dx\\ &=\int_0^\infty x(1.4e^{-2x}+0.9e^{-3x})dx\\ &=\Big[x\big(1.4\frac{e^{-2x}}{-2}+0.9\frac{e^{-3x}}{-3}\big)-\big(1.4\frac{e^{-2x}}{4}+0.9\frac{e^{-3x}}{9}\big)\Big]_0^\infty\\ &=\frac{1.4}{4}+\frac{0.9}{9}=\frac{9}{20}\end{aligned}$$

(15). Let X be a continuous r.v. with p.d.f.

$$f(x)=\begin{cases}ax^2e^{-hx} & x\geq 0\\ 0 & x<0\end{cases}$$

with $a>0, h>0$. Then the mode of X is:

A. 0

B. 2

C. h/2

D. 2/h

Solution: A point x_0 is a mode of X if f is maximum at x_0, i.e., $f'(x_0) = 0$ and $f''(x_0) < 0$
$f'(x_0) = 0 \Rightarrow axe^{-hx_0}(2 - hx_0) = 0.$
So $x_0 = 0$ or $x_0 = \frac{2}{h}$
Also $f''(x_0) = e^{-hx_0}(-4ahx_0 + ah^2x_0^2 + 2a)$ implies $f''(0) = 2a > 0$ and $f''(2/h) = -2ae^{-2} < 0.$
Hence the mode of X is $\frac{2}{h}$.

(16). Let X be a r.v. with distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then $P[F(x) \leq \frac{1}{3}]$ will be:

A. $1/3$

B. $1/3\lambda$

C. $1 - e^{-\lambda/3}$

D. $e^{-\lambda/3}$

Solution:

$$\begin{aligned} P[F(x) \leq \frac{1}{3}] &= P[1 - e^{-\lambda x} \leq \frac{1}{3}] \\ &= P[e^{\lambda x} \leq \frac{3}{2}] \\ &= P[x \leq \frac{1}{\lambda}\log\frac{3}{2}] \\ &= F(\frac{1}{\lambda}\log\frac{3}{2}) = \frac{1}{3} \end{aligned}$$

(17). The p.d.f. of a r.v. X is given by

$$f(x) = \begin{cases} cx^{-3/2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

Then the constant c and $P[\frac{1}{9} < \frac{1}{x} < \frac{1}{4}]$ are respectively:

A. 2/3 and cannot be determined

B. 1/2 and 1/6

C. 1/2 and cannot be determined

D. 2/3 and 4/9.

Solution: We know that $\int\limits_0^\infty f(x)dx = 1 \Rightarrow \int\limits_1^\infty cx^{-3/2}dx = 1 \Rightarrow c\Big[\frac{x^{-1/2}}{-1/2}\Big]_1^\infty = 1[1.5mm] \Rightarrow -2c\Big[\frac{1}{\sqrt{x}}\Big]_1^\infty = 1[1.5mm] \Rightarrow -2c(0-1) = 1 \Rightarrow c = 1/2$

$$\begin{aligned} P[\frac{1}{9} < \frac{1}{x} < \frac{1}{4}] &= P[4 < x < 9] \\ &= \int\limits_4^9 f(x)dx \end{aligned}$$

$$= \frac{1}{2}\int_4^9 x^{-3/2}dx$$
$$= \frac{1}{2}\Big[\frac{x^{-1/2}}{-1/2}\Big]_4^9$$
$$= \frac{-2}{2}\Big[\frac{1}{\sqrt{x}}\Big]_4^9$$
$$] = \frac{1}{6}$$

(18). A continuous r.v. X with support(0,4) has the p.d.f $f(x) = \frac{1}{2} - ax$. Then the value of a must be:

A. 1

B. $\frac{1}{4}$

C. $\frac{1}{8}$

D. 0

Solution: Here $\int_0^4 f(x)dx = 1 \Rightarrow \int_1^4 (\frac{1}{2} - ax)dx = 1 \Rightarrow c\Big[\frac{x}{2} - \frac{ax^2}{2}\Big]_0^4 = 1[1.5mm] \Rightarrow 2 - 8a = 1 \Rightarrow a = 1/8$

(19). Let the joint distribution of the random vector (X, Y) be given by:[1mm]

$Y\backslash X$	0	1
1/2	0.2	0.2
2	0.1	0.1
3	0.3	0.1

Then the conditional expectation of Y given $X = 0$ is:

A. 0.8

B. 2

C. 1.2

D. 2.5

Solution: $E(Y|X=0) = \sum_y yP(Y=y|X=0)$ Here $P(X=0) = \sum_y yP(Y=y, X=0) = 0.2 + 0.1 + 0.3 = 0.6$ $P(Y=y|X=0) = \frac{P(Y=y,X=0)}{P(X=0)}$ Therefore,

$$E(Y|X=0) = \frac{1}{2}P(Y=\frac{1}{2}|X=0) + 2P(Y=2|X=0) + 3P(Y=3|X=0)$$
$$= \frac{1}{2}\frac{P(Y=\frac{1}{2}, X=0)}{P(X=0)} + 2\frac{P(Y=2,X=0)}{P(X=0)} + 3\frac{P(Y=3,X=0)}{P(X=0)}$$
$$= \frac{1}{2}\frac{0.2}{0.6} + 2\frac{0.1}{0.6} + 3\frac{0.3}{0.6}$$
$$= 2$$

(20). The joint distribution of r.v.s X and Y is given by:[1mm]

$X\backslash Y$	-2	0	2
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0

Then $P[X+Y=0]$ is:

A. 3/4

B. 1/4

C. 3/8

D. 5/8

Solution: $P[X+Y=0]=P[X=0,Y=0]+P[X=2,Y=-2]=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$

(21). The joint distribution of r.v.s X and Y is given by:

$X\backslash Y$	-1	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$

Then $P[X+Y=0]$ and $P[Y\leq 0|X=1]$ are respectively:

A. 1/8 and 2/3

B. 3/8 and 2/3

C. 3/8 and 1/3

D. 1/8 and 1/3

Solution: $P[X+Y=0]=P[X=0,Y=0]+P[X=1,Y=-1]=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$ Here $P(X=1)=1/4+1/12+1/6=1/2$

$$\begin{aligned}P[Y\leq 0|X=1]&=P[Y=-1|X=1]+P[Y=0|X=1]\\&=\frac{P(Y=-1,X=1)}{P(X=1)}+\frac{P(Y=0,X=1)}{P(X=1)}\\&=\frac{1/4}{1/2}+\frac{1/12}{1/2}\\&=\frac{2}{3}\end{aligned}$$

(22). The joint distribution of r.v.s X and Y is given below with one missing value:

$X\backslash Y$	-1	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{1}{8}$	$-$

Then the missing probability and the probability $P[X+Y=0]$ respectively are:

A. 1/4 and 1/4

B. 1/8 and 3/8

C. 1/8 and 1/4

D. 1/4 and 3/8

Solution: Let p be the missing probability. Since $\sum\sum P(x,y) = 1$, we get $1/4+1/4+1/8+1/8+1/8+p = 1$ and hence $p = 1/8$. And $P[X+Y=0] = P[X=0, Y=0] + P[X=1, Y=-1] = 1/4 + 1/8 = 3/8$.

(23). The joint distribution of r.v.s X and Y is given by:[1mm]

$X\backslash Y$	-1	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$

Then $E(X|Y=0)$ is:

A. 5/24

B. 2/5

C. 1/12

D. 1/2

Solution:

$$\begin{aligned} E(X|Y=0) &= \sum_x xP(X=x|Y=0) \\ &= 0 + P(X=1|Y=0) \\ &= \frac{P(X=1, Y=0)}{P(Y=0)} \\ &= \frac{1/12}{5/24} \quad (\because P(Y=0) = 1/8 + 1/12 = 5/24) \\ &= 2/5 \end{aligned}$$

(24). The joint p.m.f. of scores of two batsmen A and B is given in the following table:[1mm]

$A's$ score	50	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{1}{18}$
	25	$\frac{2}{18}$	$\frac{4}{18}$	$\frac{2}{18}$
	15	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{2}{18}$
		$B's$ score		

The probability that they will cross a half century partnership is :

A. 1/3

B. 5/18

C. 13/18

D. 1/9

Solution:

$$\begin{aligned} P[A+B \geq 50] &= P[A=50, B=15] + P[A=50, 25] + P[A=50, B=50] \\ &+ P[A=25, B=25] + P[A=25, B=50] + P[A=15, B=50] \\ &= 2/18 + 2/18 + 1/18 + 4/18 + 2/18 + 2/18 \\ &= 13/18 \end{aligned}$$

(25). Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2 & 0 < x+y < 1, x > 0, y > 0 \\ 0 & elsewhere \end{cases}$$

$P\Big(X+Y < \frac{1}{2}\Big)$ is

A. 1/4

B. 1/2

C. 3/4

D. 1

Solution:

$$\begin{aligned} P\Big(X+Y < \frac{1}{2}\Big) &= \iint\limits_{x+y<\frac{1}{2}} f(x,y)dxdy \\ &= \int\limits_0^{\frac{1}{2}} \int\limits_0^{\frac{1}{2}-y} 2dxdy \\ &= 2\int\limits_0^{\frac{1}{2}} (\frac{1}{2}-y)dy \\ &= 2\Big[y/2 - y^2/2\Big]_0^{\frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

(26). In the above problem, $E\Big(X/Y = \frac{1}{2}\Big)$ is

A. 1/4

B. 1/2

C. 1

D. 2

Solution:$E\Big(X|Y=\frac{1}{2}\Big) = \int\limits_x xf(x|1/2)dx$ Where $f(x|1/2) = \frac{f(x,1/2)}{\int\limits_0^{1/2} f(x,1/2)dx} = 2$ Therefore, $E\Big(X|Y=\frac{1}{2}\Big) = \int\limits_0^{1/2} 2x = \Big[x^2\Big]_0^{1/2} = 1/4$

(27). Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi} & if x^2+y^2 \leq 1 \\ 0 & otherwise \end{cases}$$

Then $P\Big(Y > max(X,-X)\Big) =$

A. 1/2

B. 1/3

C. 1/4

D. 1/6

Solution:

$$\begin{aligned} P\Big(Y > max(X,-X)\Big) &= P(Y > X, Y > -X) \\ &= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \frac{1}{\pi} dydx \\ &= 2\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \frac{1}{\pi} dydx \\ &= \frac{2}{\pi} \int_{0}^{1} \sqrt{1-x^2}dx \\ &= \frac{2}{\pi}\frac{1}{2}\Big[\arcsin(x) + x\sqrt{1-x^2}\Big]_0^1 \\ &= \frac{1}{\pi}\frac{\pi}{2} \\ &= \frac{1}{2} \end{aligned}$$

(28). Which of the following functions is a probability function of a random variable X?

A. $f(x) = \begin{cases} x(2-x) & 0 < x < 2 \\ 0 & elsewhere \end{cases}$

B. $f(x) = \begin{cases} x(1-x) & 0 < x < 1 \\ 0 & elsewhere \end{cases}$

C. $f(x) = \begin{cases} 2xe^{-x^2} & -1 < x < 1 \\ 0 & elsewhere \end{cases}$

D. $f(x) = \begin{cases} 2xe^{-x^2} & x > 0 \\ 0 & elsewhere \end{cases}$

Solution:

A. $\int_0^2 x(2-x) = \Big[x^2 - x^3/3\Big]_0^2 = 4/3$, so $f(x)$ is not a p.d.f.

B. $\int_0^1 x(1-x) = \left[x^2/2 - x^3/3\right]_0^1 = 1/6$, so $f(x)$ is not a p.d.f.

C. $\int_{-1}^{1} 2xe^{-x^2} = \int_0^1 e^{-y}dy = \left[\frac{e^{-y}}{-1}\right]_0^1 = 1 - \frac{1}{e}$, so $f(x)$ is not a p.d.f.

D. $\int_0^{\infty} 2xe^{-x^2} = \int_0^{\infty} e^{-y}dy = \left[\frac{e^{-y}]}{-1}\right]_0^{\infty} = 1$, so $f(x)$ is a p.d.f.

(29). Let X and Y be r.v.'s with $E(X^2) < \infty$. Then, can conclude that

A. $V(X) \geq V(E(X|Y))$

B. $V(X) \geq E(V(X|Y))$

C. $V(E(X|Y)) \geq E(V(X|Y))$

D. $E(V(X|Y)) \geq V(E(X|Y))$

Solution: Law of total Variance: $V(X) = E(V(X|Y)) + V(E(X|Y))$ and hence $V(X) \geq E(V(X|Y))$. Since $V(X) \geq 0$, we get $V(E(X|Y)) \geq 0$ and hence $V(X) \geq E(V(X|Y))$

(30). Let X and Y be integer valued, bounded r.v.'s. Then which of the following statements is true?

A. $E(X) = \sum_y E(X|Y=y)P(Y=y)$

B. $V(X) = \sum_y V(X|Y=y)P(Y=y)$

C. $P(X=x) = \sum_y P(X=x|Y=y)P(Y=y)$

D. $E(XY) = \sum_y yE(X|Y=y)P(Y=y)$

Solution: Since X and Y be integer valued, bounded r.v.'s, they are discrete r.v.
Then $E(X) = \sum_y E(X|Y=y)P(Y=y)$.

(31). The r.v.s X_1, X_2, X_3, X_4 are independent and their common p.m.f is given below

j	0	1	2	3
$P[X_i = j]$	0.1	0.3	0.2	0.4

If $Y = max[X_1, X_2, X_3, X_4]$, then $P[Y = 2|X_1 = X_2 = X_3 = 2]$ is

A. 0.6

B. 0

C. 0.2

D. 1.0

Solution: $P[Y = 2|X_1 = X_2 = X_3 = 2] = P[X_4 \leq 2] = 0.1 + 0.3 + 0.2 = 0.6$

(32). The r.v.s X_1 and X_2 are independent and identically distributed with common p.m.f

x	0	1	2	3
$P[X_i = x]$	0.4	0.1	0.3	0.2

Let $Y = max[X_1, X_2]$. Then $P[Y = 2|X_1 = X_2 = 2]$ is

A. 0.3

B. 0.09

C. 1.0

D. 0.8

Solution: Let $A = [Y = 2] = \{\omega \mid Y(\omega) = 2\} = \{\omega \mid \max\{X_1(\omega), X_2(\omega)\} = 2\}$
and $B = [X_1 = X_2 = 2] = \{\omega \mid X_1(\omega) = X_2(\omega) = 2\}$
Then clearly $B \subset A$ and thus $A \cap B = B$.
We have to find $P(A|B)$.
But $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

The moment generating function of X is given by $M(t) = (1-t)^{-2}$ for $t < 1$. Then $E(X^3)$ is

1. 24
2. -24
3. 4
4. 1/4

Solution:
We know that $E(X^3) = M^{'''}(0) = 24(1-t)^{-5}|_{t=0} = 24$

