1 σ-fields

Notation:-

- $A, B, C-$ sets with elements a, b, c
- $\mathcal{A}, \mathcal{B}, \mathcal{C}-$ Collection of sets with elements A, B, C.

Definition 1.1 (Monotone Sequence of Sets)**.** A sequence of sets $\{A_n\}$ is said to be monotonically increasing if $A_n \subseteq A_{n+1}$, $\forall n$ and monotonically decreasing if $A_n \supseteq A_{n+1}$, $\forall n$.

Definition 1.2 (Limit of a sequence of sets)**.**

1. Suppose (A_n) is increasing. Then $A_n \subseteq A_{n+1}, \quad \forall n$
   $\Rightarrow A_n = \bigcup\limits_{i=1}^{n} A_i$
   $\therefore \lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \bigcup\limits_{i=1}^{n} A_i = \bigcup\limits_{i=1}^{\infty} A_i$
   i.e. limit of (A_n) is the union of all $A_i's$.

2. Suppose (A_n) is decreasing. Then $A_n \supseteq A_{n+1}, \quad \forall n$
   $\Rightarrow A_n = \bigcap\limits_{i=1}^{n} A_i$
   $\therefore \lim\limits_{n\to\infty} A_n = \bigcap\limits_{i=1}^{\infty} A_i$
   i.e. limit of (A_n) is the intersection of all $A_i's$.

3. Suppose (A_n) is any sequence
   Let $B_n = \bigcap\limits_{k\geq n} A_k = \inf\limits_{k\geq n} A_k$
   $\Rightarrow B_n = A_n \bigcap A_{n+1} \bigcap A_{n+2} \bigcap .... = A_n \bigcap (\bigcap\limits_{k\geq n+1} A_k)$
   $B_n = A_n \bigcap B_{n+1}$
   $B_n \subseteq B_{n+1}$
   $\therefore (B_n)$ is increasing and hence $\lim\limits_{n\to\infty} B_n = \bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k\geq n} A_k$
   and is called the lower limit of A_n denoted by $\liminf A_n$
   $\therefore \underline{lim} A_n = \liminf A_n = \bigcup\limits_{n\geq 1} \bigcap\limits_{k\geq n} A_k$
   Let, $C_n = \bigcup\limits_{k\geq n} A_k = \sup\limits_{k\geq n} A_k$
   $C_n = A_n \cup (\bigcup\limits_{k\geq n+1} A_k) = A_n \bigcup C_{n+1}$
   $C_n \geq C_{n+1}$
   $\therefore (C_n)$ is decreasing and hence $\lim\limits_{n\to\infty} C_n = \bigcap\limits_{n=1}^{\infty} C_n = \bigcap\limits_{n\geq 1} \bigcup\limits_{k\geq n} A_k$
   and is called the upper limit of A_n denoted by $\limsup A_n$
   $\therefore \limsup A_k = \bigcap\limits_{n\geq 1} \bigcup\limits_{k\geq n} A_k$

Example 1.3 (NET/SET)**.** If $G_n = \left(0, 1 + \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $\bigcap G_n$ is:

1. closed
2. open
3. both open and closed

4. neither open nor closed

Solution:

If $0 < a \leq 1$, then $0 < a \leq 1 < 1 + \frac{1}{n}$ for all n, so $a \in G_n$ for all n.
Now if $a > 1$, then $a - 1 > 0$ and hence by Archimedean property, there exists $n \in \mathbb{N}$ such that $a - 1 > \frac{1}{n} \Rightarrow a > 1 + \frac{1}{n}$, hence $a \notin G_n$.
$\therefore \cap G_n = (0, 1]$
Hence it is neither open nor closed.

Example 1.4 (NET/SET)**.** In $\mathbb{R}$, let $F_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $\forall n \in \mathbb{N}$. Then $\bigcap F_n$ is:

1. $\{0\}$
2. $\emptyset$
3. both open and closed
4. neither open nor closed

Solution:

Since $-\frac{1}{n} < 0 < \frac{1}{n} \quad \forall n \Rightarrow 0 \in F_n \; \forall n$.
If $a > 0$, then by Archimedean property, there exists $n \in \mathbb{N}$ such that $a > \frac{1}{n}$, hence $a \notin F_n$.
Similarly if $a < 0$, then $-a > 0$ and again by by Archimedean property, there exists $m \in \mathbb{N}$ such that $-a > \frac{1}{m} \Rightarrow a < -\frac{1}{m}$, hence $a \notin F_m$. $\therefore \cap F_n = \{0\}$.

Example 1.5 (NET/SET)**.** If $A_n = \left[\frac{1}{n}, 1\right]$, then $\bigcup_{n=1}^{\infty} A_n$ is:

1. $(0, 1)$
2. $(0, 1]$
3. $[0, 1]$
4. $[0, 1)$

Solution:

$$\bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$$
($\because$ if $a \leq 0$, then $a \leq 0 < \frac{1}{n}$ for all n, so $a \notin A_n$ for all n.)

Lemma 1.6. Show that $\underline{\lim} A_n \subseteq \overline{\lim} A_n$

Proof. Let $x \in \underline{\lim} A_n$

$\Rightarrow x \in \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k$

$\Rightarrow$ there exists number $n = n_0$ such that $x \in \bigcap_{k\geq n_0} A_k$

$\Rightarrow x \in A_k, \quad \forall k \geq n_0$

$\Rightarrow x \in \bigcup_{k\geq n} A_k, \quad \forall n$

$\Rightarrow x \in \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$

$\Rightarrow x \in \overline{\lim} A_n$

$\therefore \underline{\lim} A_n \subseteq \overline{\lim} A_n$ □

Example 1.7 (NET/SET)**.** A sequence of sets $\{A_n, n \geq 1\}$ converges iff :

A. $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k \subseteq \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k$

B. $\bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k \subseteq \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k$

C. $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k \subseteq \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k$

D. $\bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k \subseteq \bigcap\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k$

Solution: A_n is convergent iff $\liminf A_n = \limsup A_n$.

Since $\liminf A_n \subset \limsup A_n$, we get that

A_n is convergent iff $\limsup A_n \subset \liminf A_n$.

$\Rightarrow A_n$ is convergent iff $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k \subseteq \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} A_k$.

Exercise : If A_n and B_n are two sequences of sets then show that

1. $\overline{\lim}(A_n \bigcup B_n) = \overline{\lim} A_n \bigcup \overline{\lim} B_n$
2. $\underline{\lim}(A_n \bigcap B_n) = \underline{\lim} A_n \bigcap \underline{\lim} B_n$
3. $\overline{\lim}(A_n \bigcap B_n) \neq \overline{\lim} A_n \bigcap \overline{\lim} B_n$
4. $\underline{\lim}(A_n \bigcup B_n) \neq \underline{\lim} A_n \bigcup \underline{\lim} B_n$

Hint: let $A \neq \emptyset$. Define

$$A_n = \begin{cases} A, & \text{n is odd;} \\ \phi, & \text{n is even.} \end{cases}$$

$$B_n = \begin{cases} \phi, & \text{n is odd;} \\ A, & \text{n is even.} \end{cases}$$

Exercise : Let $A_n = \{w | 0 < w < b + \frac{(-1)^n}{n}\}, \quad b > 1$

$[i.e., A_n = (0, b + \frac{(-1)^n}{n})]$. Find lower and upper limit of A_n.

Lemma 1.8. For given any sets $A_1, A_2, \cdots, A_n$, there exists $B_1, B_2, \cdots, B_n$ of disjoint sets such that $\bigcup\limits_{i=1}^{n} B_i = \bigcup\limits_{i=1}^{n} A_i$.

Proof. We prove by induction.

Let $B_1 = A_1$ and $B_2 = A_2 \setminus B_1$. Then $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = A_1 \cup A_2$.

So assume the result is true for k sets, i.e. given $A_1, A_2, ..., A_k$, there exists $B_1, B_2, \cdots, B_k$ of disjoint sets such that $\bigcup\limits_{i=1}^{k} B_i = \bigcup\limits_{i=1}^{k} A_i$.

Define $B_{k+1} = A_{k+1} \setminus \bigcup\limits_{i=1}^{k} B_i$.

Then $B_{k+1} \cap B_i = \emptyset$ for all i and hence $B_1, B_2, \cdots, B_{k+1}$ are disjoint sets.

$$\text{And} \bigcup_{i=1}^{k+1} B_i = \bigcup_{i=1}^{k} B_i \cup B_{k+1}$$

$$= \bigcup_{i=1}^{k} A_i \cup (A_{k+1} \setminus \bigcup_{i=1}^{k} A_i)$$
$$= \bigcup_{i=1}^{k+1} A_i$$

Therefore, by induction, for given any sets $A_1, A_2, \cdots, A_n$, there exists $B_1, B_2, \cdots, B_n$ of disjoint sets such that $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. □

Definition 1.9 (Fields)**.** Let Ω be a nonempty set and $\mathcal{A}$ be a nonempty collection of subsets of Ω. Then $\mathcal{A}$ is called field (Algebra) if

1. $\mathcal{A}$ is closed under complimentation i.e. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
2. $\mathcal{A}$ is closed under union i.e. If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Example 1.10. 1. $\mathcal{A} = \{\emptyset, \Omega\}$

2. $\mathcal{A} = \mathbb{P}(\Omega)$
3. let $\Omega = \{a, b, c\}$. Then $\mathcal{A} = \{\{a\}, \{b, c\}, \Omega, \emptyset\}$

Theorem 1.11. *If $\mathcal{A}$ is a field defined on Ω, then*

1. $\phi, \Omega \in \mathcal{A}$.
2. *$\mathcal{A}$ is closed under finite union,i.e if $A_1, ..., A_n \in \mathcal{A}$, then* $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$.
3. *$\mathcal{A}$ is closed under finite intersection, i.e if $A_1, ..., A_n \in \mathcal{A}$, then* $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$.
4. *If $A, B \in \mathcal{A}$ then $A - B, B - A, A\Delta B \in \mathcal{A}$ (symmetric difference).*

Proof. 1. Since $\mathcal{A} \neq \emptyset$, let $A \in \mathcal{A}$.
Then $A^c \in \mathcal{A}$ and so $A \cup A^c = \Omega \in \mathcal{A}$.
Now as $\Omega \in \mathcal{A}$, $\Omega^c = \emptyset \in \mathcal{A}$.

2. Exer(Prove by induction).
3. Hint: use 2 for A_i^c's and Demorgan Laws.
4. Exer

□

Definition 1.12 (σ-Field)**.** Let Ω be a nonempty set and $\mathcal{A}$ be a nonempty collection of subsets of Ω. Then $\mathcal{A}$ is called σ-Field if

1. $\mathcal{A}$ is closed under complimentation i.e. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
2. $\mathcal{A}$ is closed under countable union i.e. If $A_1, A_2, ... \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Example 1.13. 1. $\mathcal{A} = \{\phi, \Omega\}$

2. $\mathcal{A} = P(\Omega)$

Exercise: Show that σ-Field is closed under countable intersection.

Example 1.14. Show that every σ-field is a field but not the converse.

Proof. Let $\mathcal{A}$ be a σ-field and $A, B \in \mathcal{A}$.

Consider $A_1 = A, A_2 = B$ and $A_i = \emptyset \, \forall i > 2$.

Then $A \cup B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Hence $\mathcal{A}$ is a field. Let $\Omega = \mathbb{N}$ and $\mathcal{A} = \{A \,|\, A \,\text{is finite or}\, A^c \,\text{is finite}\}$ We show that $\mathcal{A}$ is a field.

Let $A \in \mathcal{A}$. Then A is finite or A^c is finite.

i.e. $(A^c)^c$ is finite or A^c is finite.

$\Rightarrow A^c \in \mathcal{A}$.

Now let $A, B \in \mathcal{A}$. Then there are four possibilities:

1. Suppose A and B are finite. Then $A \cup B$ is finite and so $A \cup B \in \mathcal{A}$.
2. Suppose A is finite and B^c is finite. Then $(A \cup B)^c \subset B^c$ is finite and so $A \cup B \in \mathcal{A}$.
3. Suppose A^c is finite and B is finite. Then $(A \cup B)^c \subset A^c$ is finite and so $A \cup B \in \mathcal{A}$.
4. If A^c and B^c are finite then $(A \cup B)^c \subset A^c$ is finite and so $A \cup B \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is a field.

Now consider $A_1 = \{2\}, A_2 = \{4\}, ..., A_n = \{2n\}$ then $A_i \in \mathcal{A} \quad \forall i$.

But $\bigcup_{i=1}^{\infty} A_i = \{2, 4, 6, 8...\} \notin \mathcal{A}$, and $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \{1, 3, 5, ...\} \notin \mathcal{A}$

$\therefore \bigcup_{i=1}^{\infty} A_i \notin \mathcal{A}$. Hence $\mathcal{A}$ is not a $\sigma - field$. □

Definition 1.15 (Monotone Field)**.** A field $\mathcal{M}$ is said to be monotone field if it is closed under limit of monotone sequence in $\mathcal{M}$ i.e. if $(A_n) \subset \mathcal{M}$ such that $A_n \uparrow$ then $\lim A_n \in \mathcal{M}$ ($\bigcup A_n \in \mathcal{M}$) and if $(B_n) \subset \mathcal{M}$ such that $B_n \downarrow$ then $\lim B_n = \bigcap B_n \in \mathcal{M}$

Example 1.16. Show that $\mathcal{A} = \{A \subset \mathbb{N} \,|\, A \,\text{is finite or}\, A^c \,\text{is finite}\}$ is a field but not monotone.

Proof. By example1.14 , $\mathcal{A}$ is a field.

Now consider $A_n = \{2i \,|\, i = 1, 2, ..., n\}$.

Then each A_n is finite and hence $A_n \in \mathcal{A}$.

But $\bigcup_{i=1}^{\infty} A_i = \{2, 4, 6, 8...\} \notin \mathcal{A}$, and $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \{1, 3, 5, ...\} \notin \mathcal{A}$

$\therefore \bigcup_{i=1}^{\infty} A_i \notin \mathcal{A}$. Hence $\mathcal{A}$ is not a monotone field. □

Theorem 1.17. *Every $\sigma-$ field is monotone and conversely.*

Proof. Let $\mathcal{A}$ be a $\sigma-$ field. Let $A_n \downarrow$ then $lim A_n = \bigcap A_n$. Since $A_n \in \mathcal{A}$ and $\mathcal{A}$ is a $\sigma-$ field, $\bigcap A_n \in \mathcal{A}$ ($\because \sigma-$ field is closed under countable intersection)

$\Rightarrow \lim A_n \in \mathcal{A}$

$(B_n)\uparrow$, $B_n \in \mathcal{A}$. Since $\mathcal{A}$ is a $\sigma-$ field, $\bigcup B_n \in \mathcal{A}$,

Since $(B_n)\uparrow$, $\lim B_n = \bigcup B_n \in \mathcal{A}$

$\therefore \mathcal{A}$ is monotone field.

Conversely, let $\mathcal{A}$ be monotone and $A_i \in \mathcal{A}, i = 1, 2, ...$

Let $B_n = \bigcup\limits_{k=1}^{n} A_k$. Then $B_n \in \mathcal{A}$ ($\because A$ is field) and B_n is increasing.

$\therefore \bigcup\limits_{k=1}^{\infty} A_k = \bigcup\limits_{k=1}^{\infty} B_k \in \mathcal{A}$

$\therefore \mathcal{A}$ is a $\sigma-$ field. □

Lemma 1.18. Let $\mathcal{C}$ be a collection of subsets of Ω(need not be a field) and

$\sigma(\mathcal{C}) = \bigcap\{\mathcal{A}_i | \mathcal{A}_i \text{ is a } \sigma-\text{field such that } \mathcal{A}_i \supset \mathcal{C}\}$ [i.e., $\sigma(\mathcal{C})$ is the intersection of all $\sigma-$ fields containing $\mathcal{C}$] then $\sigma(\mathcal{C})$ is a $\sigma-$ field.

(Therefore, for every collection $\mathcal{C}$, there exists a $\sigma - field$ containing $\mathcal{C}$.)

Proof. Hint:$A \in \sigma(\mathcal{C}) = \bigcap \mathcal{A}_i$

$A \in \mathcal{A}_i, \quad \forall i$

$A^c \in \mathcal{A}_i, \quad \forall i$

$A^c \in \bigcap \mathcal{A}_i = \sigma(\mathcal{C})$. Similarly, prove for countable union. □

Definition 1.19 (Minimal $\sigma-$ field or generated $\sigma-$ field)**.** Let $\mathcal{C}$ be a collection of subsets of Ω. Then the smallest $\sigma-$ field containing $\mathcal{C}$ or $\sigma-$ field generated by $\mathcal{C}$ is denoted by $\sigma(\mathcal{C})$ and is given by

$\sigma(\mathcal{C}) = \bigcap\{\mathcal{A}_i | \mathcal{A}_i \text{ is a } \sigma-\text{field such that } \mathcal{A}_i \supset \mathcal{C}\}$.

Remark 1.20.

1. $\mathcal{C}$ always contained in $\sigma(\mathcal{C})$ i.e. $\mathcal{C} \subset \sigma(\mathcal{C})$
2. if $\mathcal{A}$ is $\sigma - field$ containing $\mathcal{C}$ then $\sigma(\mathcal{C}) \subset \mathcal{A}$ as $\sigma(\mathcal{C})$ is the smallest.

Theorem 1.21.

1. *If* $\mathcal{C}_1 \subset \mathcal{C}_2$ *then* $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$.
2. *If* $\mathcal{C}$ *is a* $\sigma-$ *field then* $\sigma(\mathcal{C}) = \mathcal{C}$.
3. $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$

Proof.

1. Given that, $\mathcal{C}_1 \subset \mathcal{C}_2$
   Since $\mathcal{C}_2 \subset \sigma(\mathcal{C}_2) \Rightarrow \mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$
   $\sigma(\mathcal{C}_2) \supset \mathcal{C}_1$ and $\sigma(\mathcal{C}_1)$ is smallest implies $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$
2. Clearly $\mathcal{C} \subset \sigma(\mathcal{C})$.
   As $\mathcal{C} \subset \mathcal{C}$ and $\mathcal{C}$ is a $\sigma-$ field $\Rightarrow \sigma(\mathcal{C}) \subset \mathcal{C}$
   $\therefore \sigma(\mathcal{C}) = \mathcal{C}$
3. Since $\sigma(\mathcal{C})$ is a $\sigma-$ field, we get $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$

□

Definition 1.22 (Borel $\sigma-$ field)**.** Let $\Omega = \mathbb{R}$ and

$\mathcal{C} = \{(a, b)| - \infty < a \le b < \infty, \quad a, b \in \mathbb{R}\}$.

Then the $\sigma-$ field generated by $\mathcal{C}$ is called the Borel $\sigma-$ field and is denoted by $\mathcal{B}$

Definition 1.23 (Borel Set)**.** Any element from the Borel $\sigma-$ field $\mathcal{B}$ is called a Borel set.

Example 1.24.

1. Every singleton set is a Borel set.
   Let $a \in \mathbb{R}$ then
   $(a - \frac{1}{n}, a + \frac{1}{n}) \in \mathcal{B}, \quad \forall n \geq 1$
   Since $\mathcal{B}$ is a $\sigma-$ field $\bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) \in \mathcal{B} \Rightarrow \{a\} \in \mathcal{B}$.
   $\therefore$ Every singleton set is Borel set.

2. Every finite set is a Borel set.
   Let A be a finite set, say $A = \{x_1, x_2..., x_n\}$ then $A = \bigcup\limits_{i=1}^{n} \{x_i\} \in \mathcal{B}$
   $\therefore A$ is a Borel set.

3. Every countable set is Borel.
   Let B be a countable set, say $B = \{x_1, x_2...\}$ then $B = \bigcup\limits_{i=1}^{n} \{x_i\} \in \mathcal{B}$
   $\therefore B$ is a Borel set. In particular, $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are Borel.

4. Every open set is Borel as any open set is the countable union of disjoint open intervals.

5. Every closed set is Borel as it is the complement of open set.

Theorem 1.25. *Let* $\mathcal{C}_1 = \{(a, b) | a \leq b, a, b \in \mathbb{R}\}$ *and* $\mathcal{C}_2 = \{(-\infty, a] | a \in \mathbb{R}\}$ *then* $\mathcal{B} = \sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$.

Proof. To show $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$, It is enough to show that $\mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$.
Let $a, b \in \mathbb{R}$. We will show that $(a, b) \in \sigma(\mathcal{C}_2)$.
Since $a, b \in \mathbb{R}$, $(-\infty, a], (-\infty, b] \in \sigma(\mathcal{C}_2)$.
As $(-\infty, a] \in \sigma(\mathcal{C}_2), (-\infty, a]^c = (a, \infty) \in \sigma(\mathcal{C}_2)$.
Then $(a, \infty) \bigcap (-\infty, b] \in \sigma(\mathcal{C}_2)$
$\Rightarrow (a, b] \in \sigma(\mathcal{C}_2) \quad \forall a, b \in \mathbb{R}$
$\Rightarrow (a, b - \frac{1}{n}] \in \sigma(\mathcal{C}_2), \forall n \geq 1$
$\Rightarrow \bigcup\limits_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{C}_2)$
$\Rightarrow (a, b) \in \sigma(\mathcal{C}_2)$
$\therefore \mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$
$\Rightarrow \sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$
To show $\mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$, let $a \in \mathbb{R}$.
Then $(a, a + n) \in \sigma(\mathcal{C}_1), \forall n$
$\Rightarrow \bigcup\limits_{n \geq 1} (a, a + n) \in \sigma(\mathcal{C}_1)$
$\Rightarrow (a, \infty) \in \sigma(\mathcal{C}_1)$
$\Rightarrow (-\infty, a] = (a, \infty)^c \in \sigma(\mathcal{C}_1) \therefore \mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$
$\Rightarrow \sigma(\mathcal{C}_2) \subset \sigma(\mathcal{C}_1)$
$\therefore \sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$ □

Theorem 1.26. *Consider* $\mathcal{C}_1 = \{(a, b) | a \leq b, a, b \in \mathbb{R}\}$
$\mathcal{C}_2 = \{[a, b) | a \leq b, a, b \in \mathbb{R}\}$, $\mathcal{C}_3 = \{(a, b] | a \leq b, a, b \in \mathbb{R}\}$

$\mathcal{C}_4 = \{[a, b] | a \leq b, a, b \in \mathbb{R}\}$, $\mathcal{C}_5 = \{(-\infty, b] | b \in \mathbb{R}\}$
$\mathcal{C}_6 = \{(-\infty, b) | b \in \mathbb{R}\}$, $\mathcal{C}_7 = \{(a, \infty) | a \in \mathbb{R}\}$
$\mathcal{C}_8 = \{[a, \infty) | a \in \mathbb{R}\}$, *then show that*
$\mathcal{B} = \sigma(\mathcal{C}_i) = \sigma(\mathcal{C}_j)$ *for all* $i \neq j$.

Proof. Exercise. □

Remark 1.27. In view of the above theorem, Borel $\sigma-$ field is the $\sigma-$ field generated by the collection of all intervals of a particular type.

Theorem 1.28. *Let* $X : \Omega \to \Omega'$ *and* $\mathcal{A}$ *be the* $\sigma-$ *field in* Ω' *then* $X^{-1}(\mathcal{A})$ *is a* $\sigma-$ *field in* Ω

Proof. Let $A \in X^{-1}(\mathcal{A}) \Rightarrow A = X^{-1}(B)$, for some $B \in \mathcal{A}$.
$\Rightarrow A^c = (X^{-1}(B))^c = X^{-1}(B^c)$. Since $B \in \mathcal{A}, B^c \subset \mathcal{A}$.
$\Rightarrow A^c = (X^{-1}(B^c)) \in \mathcal{A}$
Let $A_i \in X^{-1}(\mathcal{A}) \Rightarrow A_i = X^{-1}(B_i), B_i \in \mathcal{A}$
$\bigcup A_i = \bigcup X^{-1}(B_i) = X^{-1}(\bigcup B_i)$.
Since $B_i \in \mathcal{A}, \bigcup B_i \in \mathcal{A} \Rightarrow \bigcup A_i = X^{-1}(\bigcup B_i) \in X^{-1}(\mathcal{A})$
$\therefore X^{-1}(\mathcal{A})$ is a $\sigma-$ field
Hence inverse image of $\sigma-$ field is a $\sigma-$ field. □

2 Random Variable

Definition 2.1. Let $(\Omega, \mathcal{A})$ be a measurable space. Then a function $X : (\Omega, \mathbb{A}) \to (\mathbb{R}, \mathcal{B})$ (simply $X : \Omega \to \mathbb{R}$) is called a random variable if

$X^{-1}(\mathcal{B}) \in \mathcal{A} \quad \forall \mathbf{B} \in \mathcal{B}$ i.e. inverse image of a Borel set is measurable.

Example 2.2. Let $\Omega = \{H, T\} = \{w_1, w_2\}$ and $\mathcal{A} = \{\phi, \Omega, \{H\}, \{T\}\}$.
Define $X : \Omega \to \mathbb{R}$ by $X(w) =$ no. of heads in w. Then X is a random variable.

Proof. Here $X(\{H\}) = 1$ and $X(\{T\}) = 0$. Let $\mathbf{B}$ be any Borel set.
Then $w \in X^{-1}(\mathbf{B}) \Leftrightarrow X(w) \in \mathbf{B} \Leftrightarrow 0 \,\text{or}\, 1 \in \mathbf{B}$

$$\therefore X^{-1}(\mathbf{B}) = \begin{cases} \phi, & 0, 1 \notin \mathbf{B}; \\ \{T\}, & 0 \in \mathbf{B}, 1 \notin \mathbf{B}; \\ \{H\}, & 0 \notin \mathbf{B}, 1 \in \mathbf{B}; \\ \{H, T\}, & 0, 1 \in \mathbf{B}. \end{cases}$$

$\Rightarrow X^{-1}(\mathbf{B}) \in \mathcal{A}$ and hence X is a random variable. □

Example 2.3. Let $\Omega = \{HH, HT, TH, TT\} = \{w_1, w_2, w_3, w_4\}$ and $\mathcal{A} = P(\Omega)$. Define $X : \Omega \to \mathbb{R}$ by $X(w) =$ no. of heads in w. Then X is a random variable.

Proof. Here $X(\{HH\}) = 2$, $X(\{HT\}) = 1$, $X(\{TH\}) = 1$, $X(\{TT\}) = 0$ and thus X takes values 0,1, and 2.
Let $\mathbf{B}$ be the Borel set.
Then $w \in X^{-1}(\mathbf{B}) \Leftrightarrow X(w) \in \mathbf{B} \Leftrightarrow 0 \,\text{or}\, 1 \,\text{or}\, 2 \in \mathbf{B}$

$$\text{Hence } X^{-1}(\mathbf{B}) = \begin{cases} \phi, & 0, 1, 2 \notin \mathbf{B}; \\ \{HH\}, & 0, 1 \notin \mathbf{B}, 2 \in \mathbf{B}; \\ \{HT\}, & 0, 2 \notin \mathbf{B}, 1 \in \mathbf{B}; \\ \{TH\}, & 0, 2 \notin \mathbf{B}, 1 \in \mathbf{B}; \\ \{TT\}, & 0, 1 \notin \mathbf{B}, 0 \in \mathbf{B}; \\ \{\Omega\}, & 0, 1, 2 \in \mathbf{B}. \end{cases}$$

$\Rightarrow X^{-1}(\mathbf{B}) \in \mathcal{A}$
$\Rightarrow X$ is a random variable. □

Remark 2.4.

1. The condition for random variable depends on the measurable space $(\Omega, \mathcal{A})$ i.e. X may be a random variable on $(\Omega, \mathcal{A})$ but may not be a random variable on $(\Omega, \mathcal{A}^{'})$. For example, in Example 2.2, X is a random variable on $\mathcal{A} = P(\Omega)$ but X is not a random variable on $\mathcal{A}^{'} = \{\phi, \Omega\}$ because for $\mathbf{B} = \{0\}, X^{-1}(\mathbf{B}) = \{T\} \notin A^{'}$.

2. But in most of the cases, we consider the measurable space as $(\Omega, P(\Omega))$. In this case any real valued function(i.e.any function $X : \Omega \to \mathbb{R}$) is a random variable because, for any Borel set $B \in \mathbf{B}$, we have $X^{-1}(B) \subset \Omega \Leftrightarrow X^{-1}(B) \in P(\Omega)$.
   Hence For any function X in which every outcomes has assigned a real number is called a random variable.

Definition 2.5. Let $A \subset \Omega$. Then the indicator function or the characteristic function A is denoted by I_A or χ_A and is given by $I_A : \Omega \to \mathbb{R}$

$$I_A(w) = \begin{cases} 1, & w \in A; \\ 0, & w \notin A. \end{cases}$$

Theorem 2.6. *Let $(\Omega, \mathcal{A})$ be a measurable space and $A \subset \Omega$. Then $I_A : \Omega \to \mathbb{R}$ is a random variable iff $A \in \mathcal{A}$ (i.e. A is a measurable set).*

Proof. Let **B** be a Borel set. Then

$$I_A^{-1}(\mathbf{B}) = \begin{cases} \phi, & 0, 1 \notin \mathbf{B}; \\ A, & 1 \in \mathbf{B}, 1 \notin \mathbf{B}; \\ A^c, & 0 \in \mathbf{B}, 1 \notin \mathbf{B}; \\ \Omega, & 0, 1 \in \mathbf{B}. \end{cases}$$

$\therefore I_A$ is a random variable $\Leftrightarrow I_A^{-1}(B) \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}$. □

Theorem 2.7. *Let $(\Omega, \mathcal{A})$ be a measurable space then $X : \Omega \to \mathbb{R}$ is a random variable iff $X^{-1}((-\infty, x]) \in \mathcal{A}$ $\forall x \in \mathbb{R}$.*

Proof. Suppose X is a random variable and let $x \in \mathbb{R}$.
Since $(-\infty, x] \in \mathcal{B}$ and X is random variable, we get $X^{-1}((-\infty, x]) \in \mathcal{A}$
Conversely, suppose $X^{-1}((-\infty, x]) \in \mathcal{A} \quad \forall x \in \mathbb{R}$.
Let $\mathcal{C} = \{(-\infty, x] | x \in \mathbb{R}\}$
Then $\sigma(c) = \mathcal{B}$, the Borel $\sigma-$ field.
As $X^{-1}((-\infty, x]) \in \mathcal{A}$, we get $X^{-1}(\mathcal{C}) \subset \mathcal{A}$
$\Rightarrow \sigma(X^{-1}(\mathcal{C})) \subset \mathcal{A}$
$\Rightarrow X^{-1}(\sigma(\mathcal{C})) \subset \mathcal{A}$
$\Rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A}$
$\therefore$ X is a random variable. □

Theorem 2.8. *If $X : \Omega \to \mathbb{R}$ is a function and $(\Omega, \mathcal{A})$ is a measurable space, then the following are equivalent*

1. $\{\omega | X(\omega) \leqslant x\} \in \mathbb{A} \qquad \forall x \in \mathbb{R}$
2. $\{\omega | X(\omega) < x\} \in \mathbb{A} \qquad \forall x \in \mathbb{R}$
3. $\{\omega | X(\omega) \geq x\} \in \mathbb{A} \qquad \forall x \in \mathbb{R}$
4. $\{\omega | X(\omega) > x\} \in \mathbb{A} \qquad \forall x \in \mathbb{R}$.

Proof. 1)$\Rightarrow$ 2) Suppose $\{\omega | X(\omega) \leqslant x\} \in \mathbb{A} \quad \forall x \in \mathbb{R}$
i.e $X^{-1}(-\infty, x] \in \mathbb{A} \quad \forall x \in \mathbb{R}$
$\Rightarrow X^{-1}(\bigcup_{n=1}^{\infty} (-\infty, x - \frac{1}{n}]) \in \mathbb{A}$
$\Rightarrow X^{-1}(-\infty, x) \in \mathbb{A}$
$\Rightarrow \{\omega | X(\omega) < x\} \in \mathbb{A}$
2)$\Rightarrow$ 3)Suppose $\{\omega | X(\omega) < x\} \in \mathbb{A} \quad \forall x \in \mathbb{R}$

$\Rightarrow X^{-1}[(-\infty, x)] \in \mathbb{A}$

$\Rightarrow X^{-1}((-\infty, x)^c) \in \mathbb{A}$

$\Rightarrow X^{-1}([x, \infty)) \in \mathbb{A}$

$\Rightarrow \{\omega | x(\omega) \geq x\} \in \mathbb{A}$

3)$\Rightarrow$ 4) Suppose $\{\omega | x(\omega) \geq x\} \in \mathbb{A} \quad \forall x \in \mathbb{R} \Rightarrow X^{-1}([x, \infty)) \in \mathbb{A}$

$\Rightarrow X^{-1}(\bigcap_{n=1}^{\infty} [x + \frac{1}{n}, \infty)) \in \mathbb{A}$

$\Rightarrow X^{-1}(x, \infty) \in \mathbb{A}$

$\Rightarrow \{\omega | X(\omega) > x\} \in \mathbb{A}$

4)$\Rightarrow$ 1) Suppose $\{\omega | X(\omega) > x\} \in \mathbb{A} \quad \forall x \in \mathbb{R}$

$\Rightarrow X^{-1}(x, \infty) \in \mathbb{A}$

$\Rightarrow X^{-1}((x, \infty)^c) \in \mathbb{A}$

$\Rightarrow \{\omega | X(\omega) \leqslant x\} \in \mathbb{A}.$ □

Remark 2.9. In view of Theorem 2.7 and Theorem 2.8, we conclude that X is a random variable iff $X^{-1}((-\infty, x]) \in \mathcal{A}$ iff $X^{-1}((-\infty, x)) \in \mathcal{A}$ iff $X^{-1}((x, \infty)) \in \mathcal{A}$ iff $X^{-1}([x, \infty)) \in \mathcal{A}$.

Definition 2.10. Let $(\Omega, \mathcal{A})$ be a measurable space. Then a set valued function $P : \mathcal{A} \to \mathbb{R}$ is called a probability measure if

1. $0 \leq P(A) \leq 1 \qquad \forall A \in \mathcal{A}$
2. $P(\Omega) = 1$
3. $If \quad A_i \cap A_j = \phi \quad \forall i \neq j$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

and the space $(\Omega, \mathcal{A}, P)$ is called a probability space.

Theorem 2.11 (Properties of Probability Measure). *1.* $P(\emptyset) = 0$

2. If $A \subset B$ *then* $P(A) \leq P(B)$ *(i.e.*P *is monotone).*

3. if $\quad A_i \cap A_j = \phi \quad \forall i \neq j$*, then*
$P(\bigcup_{i=1}^{n} (A_i) = \sum_{i=1}^{n} P(A_i)$.

4. If $\{A_i\} \subset \mathbf{A}$ *(need not be disjoint) then* $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ *(i.e.* P *is sub additive).*

5. P *is continuous i.e. whenever* $(A_n) \to A$ *then* $P(A_n) \to P(A)$.

We show that every random variable induces a probability measure on Borel σ- field.

Theorem 2.12. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space. If* $X : \Omega \to \mathbb{R}$ *is a random variable, then* $P_X : \mathcal{B} \to \mathbb{R}$ *given by* $P_X(B) = P(X^{-1}(B))$ *is a probability measure on* $\mathcal{B}$.

Proof. Here $P_X : \mathcal{B} \to \mathbb{R}$ given by $P_X(B) = P(X^{-1}(B))$ is well defined because for any $B \in \mathcal{B}$, $X^{-1}(B) \in \mathcal{A}$ and hence $P(X^{-1}(B)) \in \mathcal{A}$.

1. $P_X(B) = P(X^{-1}(B)) \geq 0 \quad \forall B \in \mathcal{B}$

2. $P_X(\mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$

3. Let $B_i \subset \mathcal{B}$ such that $B_i \cap B_j = \emptyset, \quad \forall i \neq j$

   Then $X^{-1}(B_i) \cap X^{-1}(B_j) = \emptyset, \quad \forall i \neq j.$

$$\text{Hence } P_X(\bigcup_{i=1}^{\infty} B_i) = P(X^{-1}(\bigcup_{i=1}^{\infty} B_i))$$
$$= P(\bigcup_{i=1}^{\infty} X^{-1}(B_i))$$
$$= \sum_{i=1}^{\infty} P(X^{-1}(B_i)$$
$$= \sum_{i=1}^{\infty} P_X(B_i)$$

   Therefore, P_X is a probability measure on $\mathcal{B}$.

□

Definition 2.13. A function $F : \mathbb{R} \to \mathbb{R}$ is said to be a distribution function if it satisfies the following conditions:

1. F is nondecreasing i.e. if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$.

2. F is right continuous at every $x \in \mathbb{R}$ i.e. $\lim_{h\to 0} F(x+h) = F(x)$ (or $\lim_{n\to\infty} F(x + \frac{1}{n}) = F(x)$).

3. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

Remark 2.14. 1. Denote $\lim_{x\to\infty} F(x)$ by $F(\infty)$ and $\lim_{x\to-\infty} F(x)$ by $F(-\infty)$

   i.e. $F(\infty) = \lim_{x\to\infty} F(x)$ and $F(-\infty) = \lim_{x\to-\infty} F(x)$

   hence (3) is same as writing $F(\infty) = 1$ and $F(-\infty) = 0$.

2. If F is a distribution function then $0 \leq F(x) \leq 1 \quad \forall x \in \mathbb{R}$.

   proof: For any $x \in \mathbb{R}$, we have $x - n < x < x + n$ for every $n \geq 1$.

   $\Rightarrow F(x-n) \leq F(x) \leq F(x+n)$ for every $n \geq 1$

   $\Rightarrow \lim_{n\to\infty} F(x-n) \leq F(x) \leq \lim_{n\to\infty} F(x+n)$

   $\Rightarrow 0 \leq F(x) \leq 1 \quad (\because x - n \to -\infty \, and \, x + n \to \infty)$

Example 2.15. Show that the following function is a distribution function $F(x) = \begin{cases} 0, & x < 0; \\ x, & 0 \leq x < \frac{1}{2}. \\ 1, & x \geq \frac{1}{2}. \end{cases}$

Proof. 1. Since $F'(x) = 1 > 0$ in the interval $[0, \frac{1}{2})$, we get that F is non decreasing.

2. Clearly F is continuous except at $x = \frac{1}{2}$.

   Since $\lim_{h\to 0^+} F(\frac{1}{2} + h) = \lim_{h\to 0^+}(1) = 1 = F(\frac{1}{2})$, we get that F is right continuous at $x = \frac{1}{2}$.

3. $\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(1) = 1$

   And $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} 0 = 0$.

Therefore, F is a distribution function. □

Example 2.16. Show that the following function is a distribution function $F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$

Proof. 1. Since $F'(x) = e^{-x} > 0$ in the interval $[0, \infty)$, we get that F is non decreasing.

2. Clearly F is continuous except at $x = 0$.
   Since $\lim_{h \to 0^+} F(0 + h) = \lim_{h \to 0^+}(1 - e^{-h}) = 0 = F(0)$, we get that F is right continuous at $x = 0$.

3. $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty}(1 - e^{-x}) = 1$
   And $\lim\limits_{x\to-\infty} F(x) = \lim\limits_{x\to-\infty} 0 = 0.$

Therefore, F is a distribution function. □

Example 2.17. Show that the following function is not a distribution function $F(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$

Proof. Since $F'(x) = -e^{-x} < 0$ in the interval $[0, \infty)$, F is a decreasing function and hence not a distribution function. □

Exercise: Check whether the following functions are distribution functions or not.

1. $F(x) = \begin{cases} e^{x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$

2. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 + x)e^{-x}, & x \geq 0 \end{cases}$

3. $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

4. $f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$

5. $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

6. $f(x) = \begin{cases} 0, & x < 1 \\ (x - 1)^2/8, & 1 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$

Remark 2.18. 1. Sum of distribution functions and scalar times a distribution function need not be a distribution function, i.e., if F and G are distribution functions and $a \in \mathbb{R}$, then $F + G$ and aF need not be a distribution function. For Example,
$F(x) = G(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 0 \end{cases}$
Then F and G are distribution functions, but
$(F + G)(x) = \begin{cases} 0 & x \leq 0 \\ 2 & x \geq 0 \end{cases}$

is not a distribution function.

Clearly aF is not a distribution function if $a \neq 1$ as $(aF)(\infty) = a(F(\infty)) = a$.

2. Convex combination of distribution functions is a distribution function.

   i.e. Let F_i be distribution functions and $0 \leq \lambda_i \leq 1$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then the convex combination of distribution functions $\sum_{i=1}^{n} \lambda_i F_i$ is also a distribution function.

3. If F and G are distribution functions then $F.G$ is a distribution function. In particular if F is a distribution function, then F^n $n \geq 1$ is also a distribution function.

Theorem 2.19. *Let μ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Then the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \mu((-\infty, x])$ is a distribution function.*

Proof. Here $F(x) = \mu((-\infty, x])$.

Let $x_1 < x_2$

$\Rightarrow (-\infty, x_1] \subset (-\infty, x_2]$

$\Rightarrow \mu((-\infty, x_1]) \subseteq \mu((-\infty, x_2])$ $\quad$ ($\because$ μ is monotone)

$\Rightarrow F(x_1) \subseteq F(x_2)$

$\Rightarrow F$ is nondecreasing.

$$\begin{aligned}\text{For any } x \in \mathbb{R},\ \lim_{h \to 0} F(x+h) &= \lim_{h \to 0} \mu((-\infty, x+h]) \\ &= \mu(\lim_{h \to 0}(-\infty, x+h]) \quad (\because \mu \text{ is continuous}) \\ &= \mu((-\infty, x]) \\ &= F(x).\end{aligned}$$

$\Rightarrow F$ is right continuous.

$\lim\limits_{x \to \infty} F(x) = \lim_{x \to \infty} \mu((-\infty, x]) = \mu(\mathbb{R}) = 1$

$\lim\limits_{x \to -\infty} F(x) = \lim_{x \to -\infty} \mu((-\infty, x]) = \mu(\emptyset) = 0$

$\therefore$ Every probability measure defined on Borel sigma field induces a distribution function. □

Theorem 2.20. *Every real valued random variable X defined on $(\Omega, \mathcal{A}, P)$ induces a distribution function F_X.*

Proof. First we show that every real valued X induces a probability measure

$P_X : \mathcal{B} \to \mathbb{R}$ given by $P_X(B) = P(X^{-1}(B))$

Now, we show that this P_X induces a distribution function

$F_X : \mathbb{R} \to \mathbb{R}$ given by $F_X(x) = P_X((-\infty, x])$. □

Definition 2.21. Let X be a random variable defined on $(\Omega, \mathcal{A}, P)$. Then the function $F_X : \mathbb{R} \to \mathbb{R}$ defined by $F_X(x) = P(X \leq x)$ is called the distribution function or commutative distribution function of a random variable X. We denote F_X by F i.e. $F(x) = P(X \leq x)$.

Definition 2.22. Let X be a random variable and F be its distribution function. Then $J : \mathbb{R} \to \mathbb{R}$ define by $J(x) = F(x) - F(x^-)$ is called the jump of F at x, where $F(x^-)$ is the left hand limit of F at x

i.e $F(x^-) = \lim\limits_{h \to 0} F(x-h)$ or $F(x^-) = \lim\limits_{n \to \infty} F(x - \frac{1}{n})$.

Remark 2.23. 1. $J(x) \geq 0$ as $F(x^-) \leq F(x)$.

2. $P(X = x) = J(x) = F(x) - F(x^-) = F(x^+) - F(x^-)$

3. A distribution function F is continuous at x iff $J(x) = 0$ and hence discontinuous at x iff $J(x) > 0$.

4. The set of discontinuous point of distribution function F is countable.

5. The set of continuous point of F is dense in $\mathbb{R}$.

Example 2.24. 1. $F(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ x, & 0 \leq x < \frac{1}{2}; \\ 1, & x \geq \frac{1}{2}. \end{cases}$

Then F is discontinuous at $x = \frac{1}{2}$ because, $F(\frac{1}{2}^-) = \lim\limits_{h \longrightarrow 0} F(\frac{1}{2} - h)$
$= \lim\limits_{h \to 0} \frac{1}{2} - h = \frac{1}{2} \neq 1 = F(\frac{1}{2})$.
Jump of F at $\frac{1}{2} = J(\frac{1}{2}) = F(\frac{1}{2}) - F(\frac{1}{2}^{-1}) = 1 - \frac{1}{2} = \frac{1}{2}$
$\therefore P(X = \frac{1}{2}) = J(\frac{1}{2}) = \frac{1}{2}$

2. $f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$
Then $P(X = 0) = J(0) = F(0^+) - F(0^-) = 1 - 0 = 1$.

Theorem 2.25. *Let F be a distribution function of a r.v. of X. Then*

1. $P(a < X \leq b) = F(b) - F(a)$.

2. $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$.

3. $P(a < X < b) = F(b) - F(a) - P(X = b)$.

4. $P(X > a) = 1 - F(a)$.

Proof. 1. Let $A = [x \leq b]$ and $B = [x \leq a]$. Then $A \setminus B = [a < X \leq b]$

$$\begin{aligned} P(a < x \leq b) &= P(A \setminus B) = P(A) - P(B) \\ &= P(X \leq b) - P(X \leq a) \\ &= F(b) - F(a). \end{aligned}$$

2. $P(a \leq X \leq b) = P(X = a) + P(a < X \leq b) = P(X = a) + F(b) - F(a)$.

3. $P(a < X < b) = P(a < X \leq b) - P(X = b) = F(b) - F(a) - P(X = b)$.

4. $P(X > a) = P(\Omega \setminus (X \leq a)) = P(\Omega) - P(X \leq a) = 1 - F(a)$.

□

Example 2.26. Suppose the r.v. X has distribution function
$F(x) = \begin{cases} 0, & x \leq 0; \\ 1 - e^{-x^2}, & x > 0. \end{cases}$
What is the probability that X exceeds 1.

Proof. We have to find $P(X > 1)$.
But $P(X > 1) = 1 - F(1) = 1 - (1 - 1/e) = 1/e$. □

3 NET/SET questions

1. **[Jan 2006, SET]** Let $(\Omega, \mathcal{A}, P)$ be the probability space under consideration and let A_1 and A_2 be two events. Then

   A. To decide whether A_1 and A_2 are independent we need to refer to P.

   B. To decide whether A_1 and A_2 are mutually exclusive we need to refer to P.

   C. To decide whether A_1 and A_2 are exhaustive we need to refer to P.

   D. To verify that $\mathcal{A}$ is a σ-field we need to refer to P.

   Solution:

   A. A_1 and A_2 are independent if $P(A_1 \cap A_2) = P(A_1)P(A_2)$, hence we need to refer to P.

   B. A_1 and A_2 are mutually exclusive if $A_1 \cap A_2 = \emptyset$, so no need to refer to P.

   C. A_1 and A_2 are exhaustive if $A_1 \cup A_2 = \Omega$, so no need to refer to P.

   D. $\mathcal{A}$ is a σ-field if it is closed under finite union and complement, so no need to refer to P.

2. **[Aug. 2006(SET)]** Let F and G be two cumulative distribution functions. Which of the following need not be a cumulative distribution function?

   A. $\frac{1}{8}F(x) + \frac{7}{8}G(x)$

   B. $F(x)^{3/2}$

   C. $2F(x) - G(x)$

   D. $[F(x) + G(x)]^2/4$

   Solution:

   A. Convex combination of distribution functions is a distribution function

   B. Positive power of a distribution function is a distribution function

   C. $2F(x) - G(x)$ need not be a distribution function

   D. Convex combination of distribution functions is a distribution function

3. **[Aug. 2006(SET)]** The cumulative distribution function of a random variable X is given by $F_X(x) = \begin{cases} 0, & x < -2; \\ \frac{5-|x|}{8}, & -2 \leq x < -1. \\ \frac{8-|x|}{8}, & -1 \leq x < 0; \\ 1, & x \geq 0; \end{cases}$
   Hence the points of discontinuous of F_X are

   A. $\{-2, -1, 0\}$

   B. $\{-1, 0\}$

   C. $\{-2, -1\}$

D. $\{-2, 0\}$

Solution:

F_X can be discontinuous at most at -2,-1 and 0.

Since $F(0) = F(0^-) = 1$ and $F(-2) \neq F(-2^-)$, and $F(-1) \neq F(-1^-)$, we get that the set of points of discontinuous of F_X is $\{-2, -1\}$.

4. Let $A_1, A_2, ...$ be a sequence of events on a probability space and let $A = \bigcup\limits_{k=1}^{\infty} A_k$. Which of the following statements is not always true?

A. $P(A) = \sum\limits_{k=1}^{\infty} P(A_k)$

B. $P(A^c) \leq P(A_1^c)$

C. $P(A) = P(\lim\limits_{n\to\infty} \bigcup\limits_{k=1}^{n} A_k)$

D. $P(A^c) \geq 1 - \sum\limits_{k=1}^{\infty} P(A_k)$

Solution:

$P(A) = \sum\limits_{k=1}^{\infty} P(A_k)$ is true for the disjoint sets but not true always.

5. **[Feb 2007(SET)]** Let F and G be (probability) distribution functions. Which of the following is not a (probability) distribution function?

A. $F(x)G(x)$

B. $\frac{11}{10}F(x) - \frac{1}{10}G(x)$

C. $\frac{3}{5}F(x) + \frac{2}{5}G(x)$

D. $\frac{1}{6}F(x) + \frac{5}{6}G^3(x)$

Solution:

Convex combination and product of distribution functions is again a distribution function.

$\frac{11}{10}F(x) - \frac{1}{10}G(x)$ need not be a distribution function.

6. **[Jan. 2009(SET)]** The distribution function of a random variable X is given below

$$F(x) = \begin{cases} 0, & x < 0; \\ \frac{1}{4}, & 0 \leq x < 1; \\ \frac{1}{2} + \frac{1}{2}(1 - exp{-}(x-1)), & x \geq 1; \end{cases}$$

Then the probabilities $P[0 \leq X < 1]$ and $P[\frac{1}{2} < X \leq 1]$ respectively are

A. 0 and $\frac{1}{4}$

B. $\frac{1}{4}$ and $\frac{1}{2}$

C. $\frac{1}{4}$ and $\frac{1}{4}$

D. 0 and $\frac{1}{2}$

Solution:

$P[0 \leq X < 1] = F(1) - F(0) + P(X = 0) - P(X = 1) = 1/2 - 1/4 = 1/4$

$P[\frac{1}{2} < X \leq 1] = F(1) - F(1/2) = 1/2 - 1/4 = 1/4$

7. **[Dec 2013] SET** Let $F : \mathbb{R} \to [0,1]$ be the distribution function of a random variable X. Which of the following statements is true?

   A. F has atmost finite number of discontinuity points

   B. If F_1 and F_2 are two distribution functions, then $F_1 + F_2$ is also a distribution function

   C. $F : \mathbb{R} \to \mathbb{R}$ defined by
   $$F(x) = \begin{cases} 0, & x \leq 0; \\ \alpha + \beta e^{-x^2/2}, & x > 0. \end{cases}$$
   is a distribution function for each value of (α, β)

   D. F is a non-decreasing function. **Solution:**

   Distribution function can have infinite number of discontinuous points, but countable.

   Sum of distribution functions need not be a distribution function.

   But if F is a distribution function, then it is non-decreasing.

