1 Discrete Uniform Random Variable

Definition 1.1. Discrete Uniform Random Variable

A discrete random variable X is said to follow discrete uniform distribution if all its n outcomes, say $x_1, x_2, ..., x_n$ are equally likely i.e. with equal probability. And its p.m.f. is given by $p(x_i) = P(X = x_i) = \frac{1}{n}$ $\forall i = 1, 2, 3, \ldots, n$ where n is called parameter of this distribution and is denoted by $X \sim DU(n)$.

Expectation of X is

$$\begin{aligned} E(X) &= \sum xP(X = x) \\ &= \sum_{x=1}^{n} x\frac{1}{n} \\ &= \frac{1}{n}\frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

$$\begin{aligned} And\ E(X^2) &= \sum_{x=1}^{n} x^2 P(X = x) \\ &= \sum_{x=1}^{n} x^2 \frac{1}{n} \\ &= \frac{1}{n}\sum_{x=1}^{n} x^2 \\ &= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - E^2(X) \\ &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\ &= \frac{n+1}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right) \\ &= \frac{n+1}{2}\left(\frac{4n+2-3n-3}{6}\right) \\ &= \frac{n+1}{2}\left(\frac{n-1}{6}\right) \\ &= \frac{n^2-1}{12} \end{aligned}$$

Moment Generating Function of X is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=1}^{n} e^{tx} P(X = x) \\ &= \sum_{x=1}^{n} \frac{1}{n} e^{tx} \end{aligned}$$

$$= \frac{1}{n}\sum_{x=1}^{n} e^{tx}$$
$$= \frac{1}{n}(e^t + e^{2t} + ... + e^{nt})$$
$$= \frac{1}{n}\left(\frac{e^t(e^{nt}-1)}{e^t-1}\right)$$
$$= \frac{e^t}{n}\left(\frac{e^{nt}-1}{e^t-1}\right).$$

Remark 1.2. Using $M_X(t)$ we can find $E(X)$ and $E(x^2)$ and hence the variance.

Distribution Function

$$F(x_i) = P(X \le x_i)$$
$$= \sum_{j\le i} P(X = x_j) \quad \forall j = 1, 2, ..., i$$
$$= \sum_{j\le i} \frac{1}{n} = i\left(\frac{1}{n}\right) = \frac{i}{n}.$$

Example 1.3.
1. If X denotes the birthday of a person, then it may be either Sunday, Monday,..., Saturday with equal probability $\frac{1}{7}$. Therefore $X \sim DU(7)$.
2. If X denotes the number of heads in an outcome of tossing a single coin. Then $X = 0, 1$ with $P(X = 0) = \frac{1}{2}$ and $P(X = 1) = \frac{1}{2}$.
   $\therefore X \sim DU(2)$.
3. Let X denotes the face value of an unbiased die, then $X = 1, 2, \ldots, 6$ and $P(X = i) = \frac{1}{6} \forall i = 1, 2, \ldots, 6$. So $X \sim DU(6)$.

2 Bernoulli's random variable

A trail is called a Bernoulli trial if it has only two outcomes, denoted by S for success and F for failure with $P(S) = p$ and $P(F) = 1 - p$. If we let $X = 1$ when the outcome is success and $X = 0$ when the outcome is failure then the p.m.f. of X is given by
$P(X = 1) = p$ and $P(X = 0) = 1 - p$.

Definition 2.1. A random variable X is said to be a Bernoulli random variable if its p.m.f. is $P(X = 1) = p$ and $P(X = 0) = 1 - p$ where p is the probability that the trail is a success. It is denoted by $X \sim Bern(p)$.

Example 2.2.
1. If X denotes the number of heads in tossing a single coin, then X is a Bernoulli random variable with $p = 1/2$.
2. Consider throwing a die once. Then the random variable
   $$X(\omega) = \begin{cases} 1, & \text{face value of } \omega = 5; \\ 0, & \text{otherwise.} \end{cases}$$
   is a Bernoulli random variable with $p = 1/6$.

Expectation

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{1} xP(X=x) \\
&= 0P(X=0) + 1P(X=1) \\
&= p
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \sum_{x=0}^{1} x^2P(X=x) \\
&= 0P(X=0) + P(X=1) \\
&= p
\end{aligned}
$$

Variance

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= p - p^2 \\
&= p(1-p) \\
&= pq
\end{aligned}
$$

M.G.F.

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \sum_{x=0}^{1} e^{tx}P(X=x) \\
&= 1P(X=0) + e^tP(X=1) \\
&= (1-p) + e^tp \\
&= 1 - p(1-e^t) \\
&= 1 - p + e^tp \\
&= q + pe^t
\end{aligned}
$$

Distribution Function $F(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$

3 Binomial Random Variable

Suppose that n Bernoulli trials with probability of success p are carried out.

If X represents the number of successes that occur during the n trails, then $X = x$ represents x successes out of n trails.

One such outcome is $s.s \ldots s(x\,times).f.f \ldots f(n-x\,times)$

Probability of this outcome $= p.p\ldots p.q.q\ldots q = p^x q^{n-x}$
And there are $\binom{n}{x}$ different such outcomes each with probability $p^x q^{n-x}$.
$\therefore P(X = x) = \binom{n}{x}p^x q^{n-x} \quad x = 0, 1, 2..., n.$

Definition 3.1. A discrete random variable X is said to be binomial r.v. if its p.m.f. is given by $P(X = x) = \binom{n}{x}p^x q^{n-x} \quad x = 0, 1, 2, \ldots, n$, where p is the probability of successes in each trial. And is denoted by $X \sim Binom(n, p)$.

A binomial experiment is one that has these characteristics:

1. The experiment consists of n identical Bernoulli trials.
2. The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is equal to $(1 - p) = q$.
3. The trials are independent.
4. We are interested in x, the number of successes observed during the n trials, for $x = 0, 1, 2, ..., n$.

- Expectation

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} xP(X = x) \\
&= \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x} \\
&= \sum_{x=1}^{n} x\binom{n}{x}p^x q^{n-x} \\
x\binom{n}{x} &= \frac{xn!}{(n-x)!x!} \\
&= \frac{xn(n-1)!}{((n-1)-(x-1))!x(x-1)!} \\
&= \frac{n(n-1)!}{((n-1)-(x-1))!(x-1)!} \\
&= n\binom{n-1}{x-1} \\
\therefore E(X) &= n\sum_{x=1}^{n}\binom{n-1}{x-1}p^x q^{n-x} \\
&= n\sum_{j=1}^{n-1}\binom{n-1}{j}p^{j+1}q^{n-j-1} (Let j = x-1) \\
&= np\sum_{j=0}^{n-1}\binom{n-1}{j}p^j q^{(n-1)-j} \\
&= np(p+q)^{n-1} \\
&= np
\end{aligned}
$$

-

$$E(X^2) = \sum_{x=0}^{n} x^2 P(X = x)$$

$$
\begin{aligned}
&= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} [x(x-1)+x] \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \\
x(x-1)\binom{n}{x} &= \frac{x(x-1)n(n-1)(n-2)!}{((n-2)-(x-2))!x(x-1)(x-2)!} \\
&= \frac{n(n-1)(n-2)!}{((n-2)-(x-2))!(x-2)!} \\
&= n(n-1)\binom{n-2}{x-2} \\
E(X^2) &= \sum_{x=2}^{n} n(n-2)\binom{n-2}{x-2} p^x q^{n-x} + np \\
&= n(n-1)\sum_{j=0}^{n-2})\binom{n-2}{j} p^{j+2} q^{n-j-2} + np \\
&= n(n-1)p^2 \sum_{j=0}^{n-2})\binom{n-2}{j} p^{j} q^{(n-2)-j} + np \\
&= n(n-1)p^2 + np
\end{aligned}
$$

- <u>Variance</u>

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= np(1-p) \\
&= npq
\end{aligned}
$$

- <u>M.G.F.</u>

$$
\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum_{x=0}^{n} e^{tx} P(X=x) \\
&= \sum_{x=0}^{n} e^{tx} \binom{n}{m} p^x q^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{m} (pe^t)^x q^{n-x} \\
M_x(t) &= (pe^t + q)^n
\end{aligned}
$$

- <u>Distribution Function</u>

$$
\begin{aligned}
F(x) &= P(X=x) \\
&= \sum_{i \le x} P(X=i)
\end{aligned}
$$

$$F(x) = \sum_{i=0}^{x} \binom{n}{i} p^i q^{n-i}$$

- Recursive formula for $P(X = k+1)$ for $k = 0, 1, 2, \ldots, n-1$

$$\begin{aligned}
\frac{P[x=k+1]}{P[x=k]} &= \frac{\binom{n}{k+1} p^{k+1} q^{n-k-1}}{\binom{n}{k} p^k q^(n-k)} \\
&= \frac{p}{q} \frac{\frac{n!}{(n-k-1)!(k+1)!}}{\frac{n!}{(n-k)!k!}} \\
&= \frac{p}{q} \frac{(n-k)(n-k-1)!k!}{(n-k-1)!(k+1)k!} \\
&= \frac{p(n-k)}{q(k+1)} \\
P[x=k+1] &= \frac{p(n-k)}{q(k+1)}
\end{aligned}$$

Theorem 3.2. *If $X \sim Binom(n,p)$ then $(n-X) \sim Binom(n,q)$ where $q = (1-p)$.*

Proof. Since $X \sim Binom(n,p)$,$M_X(t) = (q + pe^t)^n$

$$\begin{aligned}
M_{(n-X)}(t) &= E(e^{tn-tX}) \\
&= e^{nt} E(e^{-tX}) \\
&= e^{nt}(q + pe^{-t})^n \\
&= (e^t(q + pe^{-t}))^n \\
&= (qe^t + p)^n
\end{aligned}$$

Therefore, $(n-X) \sim Binom(n,q)$. □

Theorem 3.3. *If X_1 and X_2 are independent binomial random variables such that X_1 $\sim Binom(n_1, p)$, X_2 $\sim Binom(n_2, p)$ then $X_1 + X_2$ $\sim Binom(n_1 + n_2, p)$.*

Proof. Because X_1 and X_2 are independent, we have

$$\begin{aligned}
M_{X_1+X_2}(t) &= M_{X_1}(t).M_{X_2}(t) \\
&= (q + pe^t)^{n_1}.(q + pe^t)^{n_2} \\
&= (q + pe^t)^{n_1+n_2}
\end{aligned}$$

$\therefore X_1 + X_2$ ~Binom$(n_1 + n_2, p)$. □

4 Geometric Distribution

The event of interest is about first success. So we carry Bernoulli trials until the first success(so the no. of trials is not fixed).
Let $X = x$ denotes the x number of failures before getting the first success. Then

$$P(X = x) = \text{Probability of } x \text{ failures}$$

$$= P(\underbrace{fff...ff}_{x-times}.s)$$
$$= (q^x)p$$

Where $x = 0, 1, 2, ...,$ and p is the probability of success and $q = p - 1$.

Definition 4.1. A discrete random variable X is said to follow Geometric distribution of its p.m.f. is given by $P(X = x) = (q^x)p$ where $x = 0, 1, 2, ..$ and p is the probability of success and $q = p - 1$.
And is denoted by $X \sim G(p)$.

Theorem 4.2. *If X is a geometric random variable, show that*

1. $E(X) = \frac{q}{p}$
2. $Var(X) = \frac{q}{p^2}$
3. $M_X(t) = \frac{q}{1-qe^t}$
4. $F(x) =$.

Proof. First we find the moment generating function of X.

$$\begin{aligned} M_X(t) &= E(e^{Xt}) \\ &= \sum_{x=0}^{\infty} e^{xt} q^x p \\ &= p \sum_{x=0}^{\infty} (e^t q)^x p \\ &= p[1 + e^t q + (e^t q)^2 + (e^t q)^3 + ...] \\ &= p[\frac{1}{1 - e^t q}] \quad \text{(it is the infinite sum of geometric series)} \end{aligned}$$

We know that $E(X) = M_X^{'}(0)$ and $E(X^2) = M_X^{''}(0)$.
But $M_X^{'}(t) = \frac{pqe^t}{(1-e^t q)^2}$.
Hence $E(X) = \frac{pq}{(1-q)^2} = \frac{q}{p}$
Also $M_X^{''}(t) = pq\frac{e^t(1-e^t q)+2qe^{2t}}{(1-e^t q)^3}$.
So $E(X^2) = \frac{q+q^2}{p^2}$
Thus $Var(X) = E(X^2) - E^2(X) = \frac{q}{p^2}$
Now the distribution function of X is

$$\begin{aligned} F(x) &= \sum_{i=0}^{x} P(X = i) \\ &= p \sum_{i=0}^{x} q^i \\ &= p[1 + q + q^2 + ... + q^x] \\ &= p(\frac{1 - q^x}{1 - q} \\ &= 1 - q^x \end{aligned}$$

□

Remark 4.3. $V(x) = \frac{1}{p}(\frac{q}{p}) = \frac{1}{p}(E(X)) \geq (E(X)) \Rightarrow (V(X)) \geq (E(X))$.

Theorem 4.4 (**Lack of Memory property**). *If X is a geometric r.v., then for any positive integers m and n, $P(X \geq (n+m)|X \geq m) = P(X \geq n)$.*

Proof. For any $k \geq 0$,

$$\begin{aligned}
P(X \geq k) &= 1 - P(x < k) \\
&= 1 - \sum_{i=0}^{k-1} P(x = i) \\
&= 1 - \sum_{i=0}^{k=1} q^i p \\
&= 1 - p\sum_{i=0}^{k-1} q^i \\
&= 1 - P\frac{(1-q^k)}{1-q} \\
&= 1 - (1 - q^k) \\
&= q^k
\end{aligned}$$

Hence

$$\begin{aligned}
P(X \geq (n+m)|X \geq m) &= \frac{P[X \geq (n+m), X \geq m]}{P(X \geq m)} \\
&= \frac{P(X \geq (n+m))}{P(X \geq m)} \\
&= \frac{q^{n+m}}{q^m} \\
&= q^n \\
&= P(X \geq n)
\end{aligned}$$

□

Remark 4.5. Lack of Memory property says that if there are m failures intially, the chance of atleast n more failures before the first success is exactly same as if we started the experiment for the first time and the information of initial m failures is not given to us.

5 Negative Binomial r.v:

The Bernuolli trials are carried out till the k^{th} success occur. Let X denotes the total no of failures before getting k^{th} success

$P(X = x)$ = probability of getting $k^(th)$ success with x failures.

One of such outcome is $\underbrace{fff\ldots f}_{x-times}\ \underbrace{s.s\ldots s}_{(k-1)times}\ .s$

Probability of this outcome is $q^x p^k$.

Since there are $x+k-1$ trials before $k^(th)$ success in which x failures and $(k-1)$ success, there are $\binom{x+k-1}{x}$ different outcomes
$\therefore P(X=x) = \binom{x+k-1}{x} q^x p^k \ x = 0, 1, 2 \ldots$
where k is the number of success and p is the probability of the success and is denoted by $X \sim NB(k, p)$.

Remark 5.1.

1. If $k = 1$ then negative binomial is same as geometric distribution.
2. In binomial distribution, we look at number of success in (n) trials, where as in negative binomial distribution, we look at the number of failures and hence the name.
3. The negative binomial series is

$$\begin{aligned}(b+a)^{-n} &= \sum_{k=0}^{\infty} \binom{-n}{k} b^k a^{-n-k} \quad |b| < a \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} b^k a^{-n-k} \\ i.e. \binom{-n}{k} &= (-1)^k \binom{n+k-1}{k}\end{aligned}$$

Hence

$$\begin{aligned}P(X=x) &= \binom{k+x-1}{x} p^k q^x \\ &= (-1)^x \binom{-k}{x} p^k q^x\end{aligned}$$

This is one of the reason to name as negative binomial.

M.G.F: We know, by the negative Binomial expansion
$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$.

$$\begin{aligned}M_X(t) &= \sum e^{tx} P(X=x) \\ &= \sum e^{tx} \binom{x+k-1}{x} p^k q^x \\ &= p^k \sum \binom{k+x-1}{x} (qe^t)^x \\ &= p^k (1-qe^t)^{-k} \quad if \quad |qe^t| < 1\end{aligned}$$

Expectation
We know that $E(X) = M_X^{'}(0)$
But $M_X^{'}(t) = kqp^k e^t (1-qe^t)^{-k-1}$
So $E(X) = M_X^{'}(0) = kqp^k (1-q)^{-k-1} = \frac{kq}{p}$
Variance
We know that $Var(X) = E(X^2) - E^2(X)$ and$E(X^2) = M_X^{''}(0)$
But $M_X^{''}(t) = kqp^k e^t (1-qe^t)^{-k-2} (1 - qe^t + q(k+1))$
So $E(X^2) = M_X^{''}(0) = \frac{k(k+1)q^2}{p^2} + \frac{kq}{p}$

Therefore, $Var(X) = E(X^2) - E^2(X) = \frac{kq}{p^2}$ (Exer: simplify in detail)

Recursive formula:

$$
\begin{aligned}
P(X=0) &= p^k \\
\frac{P[X=x+1]}{P[X=x]} &= \frac{\binom{x+1+k-1}{x+1}q^(x+1)p^k}{\binom{x+k-1}{x}q^x p^k} \\
&= \frac{\frac{(x+k)!q^x qp^k}{(x+k-x-1)!(x+1)!}}{\frac{(x+k-1)!q^x p^k}{x!(x+k-1-x)}} \\
&= \frac{(x+k)(x+k-1)!q}{(x+1)x!}\frac{x!}{(x+k-1)!} \\
&= \frac{(x+k)}{(x+1)}q \\
\therefore P[X=x+1] &= q\frac{(x+k)}{(x+1)}P[X=x] \qquad x=0,1,2\ldots
\end{aligned}
$$

6 Hypergeometric distribution

Suppose a finite population consist of N units, m out of which are of a particular type (for instance defective). Now n units are drawn from population randomly. Let $X =$ number of particular units (defective)
i.e. $X = x$ represents x particular items out of n selected
Then $P(X=x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \quad x = 0,1,2...\min\{m,n\}$
Because, the number of ways of selecting x particular items out of m is $\binom{m}{n}$ and the remaining items $n-x$ of $N-m$ is $\binom{N-m}{n-x}$.

Definition 6.1. A discrete r.v. X is said to follow hypergeometric if its p.m.f. is given by
$P(X=x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \quad x = 0,1,2...\min\{m,n\}$
and it is denoted by $X \sim H(N,m,n)$

Expectation

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\min\{m,n\}} xP(X=x) \\
&= \sum_{x=0}^{\min\{m,n\}} \frac{x\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \\
&= \frac{1}{\binom{N}{n}} \sum_{x=0}^{\min\{m,n\}} \frac{m!}{(m-x)!(x-1)!}\binom{N-m}{n-x} \\
&= \frac{m}{\binom{N}{n}} \sum_{x=0}^{\min\{m,n\}} \binom{m-1}{x-1}\binom{N-m}{n-x} \\
&= \frac{m}{\binom{N}{n}} \sum_{x=0}^{\min\{m,n\}} \binom{m-1}{x-1}\binom{(N-1)-(m-1)}{(n-1)-(x-n)} \\
&= \frac{m}{\binom{N}{n}}\binom{N-1}{n-1} \qquad (\because \binom{N}{n} = \sum_{x=1}^{\min\{m,n\}}\binom{m}{x}\binom{N-m}{n-x}) \\
&= \frac{m(N-1)!}{(N-1-n+1)!(n-1)!}\frac{(N-n)!n!}{N!}
\end{aligned}
$$

$$E(X) = \frac{mn}{N}$$

$$\begin{aligned}
E(X^2) &= \sum_{x=0} x^2 P(X=x) \\
&= \sum_x \frac{[x(x-1)+x]\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} + \frac{mn}{N} \\
&= \frac{1}{\binom{N}{n}} \sum_x \frac{m!}{(m-x)!(x-2)!}\binom{N-m}{n-x} + \frac{mn}{N} \\
&= \frac{m(m-1)}{\binom{N}{n}} \sum_x \binom{m-2}{x-2}\binom{N-2-(m-2)}{n-2-(x-2)} + \frac{mn}{N} \\
&= \frac{m(m-1)}{\binom{N}{n}} \binom{N-2}{n-2} + \frac{mn}{N} \\
&= \frac{m(m-1)(N-2)!}{(N-2-n+2)!(n-2)!}\frac{(N-n)!n!}{N!} + \frac{mn}{N} \\
E(X) &= \frac{mn(m-1)(n-1)}{N(N-1)} + \frac{mn}{N}
\end{aligned}$$

Exercise: Prove that $Var(X) = \frac{mn}{N}\frac{N-n}{N-1}\left(1-\frac{m}{N}\right)$

Remark 6.2. There is no closed form (explicit formula) for MGF of hypergeometric r.v., even though it exists.

7 Poisson Random Variable

1. Event of interest occur with small probability.
2. Event of interest is to count number of times that event occur.
3. No upper bound can be fixed for event of interest.

Some typical situations of Poisson random variable:

1. No. of accidents at a specific spot
2. No. of telephone calls received at a particular time of interval
3. No of hits to a website
4. No. of defects in a large no. of items
5. No. of infected individuals in a population
6. No. of typographical errors in a newspaper, etc

Definition 7.1. A discrete r.v. X is said to follow Poisson with parameter λ if its p.m.f. is given by $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \ldots$
and it is denoted by $X \sim P(\lambda), \quad \lambda > 0$

Expectation:

$$\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} \frac{xe^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&= e^{-\lambda}\sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!} \\
&= \lambda e^{-\lambda}\sum_{y=0}^{\infty} \frac{\lambda^{y}}{y} \\
&= \lambda e^{-\lambda}e^{\lambda} \\
E(X) &= \lambda \\
E(X^2) &= \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum \frac{xe^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} + \lambda \\
&= e^{-\lambda}\lambda^2(\sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!}) + \lambda \\
&= \lambda^2 + \lambda \\
\underline{Variance:}\ Var(X) &= E(X^2) - E^2(X) \\
&= \lambda^2 + \lambda - \lambda^2 \\
Var(X) &= \lambda
\end{aligned}$$

M.G.F:

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x=0}^{\infty} e^{tx}P(X=x) \\
&= \sum_{x=0}^{\infty} \frac{e^{tx}e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda}\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= e^{-\lambda}e^{\lambda e^t} \\
M_x(t) &= e^{\lambda(e^t-1)}
\end{aligned}$$

Exercise: If $X_i \sim P(\lambda_i)$ and are independent, then $\sum_{i=1}^{n} X_i \sim P(\sum_{i=1}^{n} \lambda_i)$.
$(Hint : M_{X_1+X_2}(t) = M_{X_1}(t).M_{X_2}(t))$

8 Degenerate Random Variable

Definition 8.1. A discrete random variable X is called degenerate if $\exists k \in \mathbb{R}$ s.t $P(X = k) = 1$ and $P(X = x) = 0$ for all $x \neq k$.

1. $E(X) = kP(X = k) + \sum_{x \neq k} 0P(X = x) = k$
2. $E(X^2) = k^2P(X = k) + \sum_{x \neq k} 0P(X = x) = k^2$
3. $Var(X) = E(X^2) - E^2(X) = k^2 - k^2 = 0$

9 NET/SET Questions

1. Suppose X has binomial $B(n, p)$ distribution. The moment generating function of X is

   A. $(p + qt)^n$

   B. $(p + qe^t)^n$

   C. $(q + pe^t)^n$

   D. $(q + pt)^n$

   Solution:

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \sum_{r=0}^{n} e^{tr} \binom{n}{r} p^r q^{n-r} \\ &= \sum_{r=0}^{n} \binom{n}{r} (pe^t)^r q^{n-r} \\ &= (q + pe^t)^n \end{aligned}$$

2. The expected number of heads in 300 tosses of a fair coin is:

   A. 300

   B. 250

   C. 200

   D. 150

   Solution: Since X follows Binomial, its expectation is $E[X] = np = 300\frac{1}{2}) = 150$.

3. The probability that a certain machine will produce a defective item is $\frac{1}{4}$. If a random sample of 8 items is taken from the output of the machine, what is the probability that there will be 7 or more defective in the sample?

   A. $\frac{25}{(256)^2}$

   B. $\frac{4}{(256)^2}$

   C. $\frac{24}{(256)^2}$

   D. $\frac{5}{(256)^2}$

Solution:
Here the random variable follows Binomial(8, $\frac{1}{4}$). Then

$$P(X=k)=\binom{8}{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{8-k}$$

Hence, the probability of 7 or more defective items is

$$P(X=7)+P(X=8)=\binom{8}{7}\left(\frac{1}{4}\right)^{7}\left(\frac{3}{4}\right)^{8-7}+\binom{8}{8}\left(\frac{1}{4}\right)^{8}\left(\frac{3}{4}\right)^{8-8}=\frac{25}{(256)^2}$$

4. The probability that a certain machine will produce a defective item is $\frac{1}{4}$. If a random sample of 6 items is taken from the output of this machine, what is the probability that there will be 5 or more defective in the sample?

A. $\frac{1}{4096}$

B. $\frac{4}{4096}$

C. $\frac{19}{4096}$

D. $\frac{18}{4096}$

Solution:
Here the random variable follows Binomial(6, $\frac{1}{4}$). Then

$$P(X=k)=\binom{6}{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{6-k}$$

Hence, the probability of 5 or more defective items is

$$P(X=5)+P(X=6)=\binom{6}{5}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{6-5}+\binom{6}{6}\left(\frac{1}{4}\right)^{6}\left(\frac{3}{4}\right)^{6-6}=\frac{19}{4096}$$

5. A survey of college students taking the professional exam to be certified as public school teacher shows that 15 percent fail. On a national exam day, 12,000 students take the test. Let X denote the number who fail. The mean value of X is:

A. 12,000

B. 1,800

C. 1,500

D. 10,200

Solution:
Here the random variable follows Binomial(12000, 0.15).
Hence, $E(X)=12000(0.15)=1800$.

6. Which of the following is not an assumption of binomial Distribution?

A. All trials must be identical

B. The probability of success in the trails is equal to 0.5

C. All trails must be independent

D. Each trail must be classified as a success or a failure.

Solution:

The probability of success can be any value between 0 and 1.

7. Let X be a random variable with mean 3.6 ad variance 6. Which of the following is not correct?

A. X can not follow a Binomial distribution

B. X can not follow a Poisson distribution

C. X can not follow an exponential distribution

D. X can not follow a normal distribution

Solution:

If X follows binomial, then mean=$np = 3.6$ and variance=$6 = np(1-p)$, implies,$p = -\frac{2}{3}$, which is not possible.
Hence it cannot follow binomial.

8. Let X be a random variable with mean equal to its variance. Then X has

A. Binomial distribution

B. Normal distribution

C. Poisson distribution

D. None of the above

Solution:

In Poisson distribution, the expectation and its variance are both equal to its parameter λ.

9. If independent binomial experiments are conducted with $n = 10$ trials. If the probability of success in each trial is $p = 0.6$, then the average number of successes per experiment is:

A. 4

B. 6

C. 8

D. 10

Solution:

Here X follows Binomial and hence $E(X) = np = 10(0.6) = 6$.

10. Let X and Y be two independent Poisson variables each with mean 1. Then $P[X + Y = 0]$ equals

A. $2e^{-2}$

B. $2e^{-1}$

C. e^{-2}

D. e^{-1}

Solution: If X and Y are independent such that $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$, then $(X+Y) \sim P(\lambda_1+\lambda_2)$. So $(X+Y) \sim P(2)$.
Therefore, $P[X+Y=0] = \frac{e^{-2}2^0}{0!} = e^{-2}$

11. Let X be a binomial r.v. with parameter $n = 7$ and $p = 0.5$. Which of the following statement is correct?

A. The r.v. $2X$ is binomial(14,0.5)

B. $X + \frac{1}{2}$ is continuous r.v.

C. $P(X = 3) = P(X = 4)$

D. The r.v. $2X$ is binomial(7,0.25).

Solution: $P(X=3) = \binom{7}{3}(0.5)^3(0.5)^4 = \binom{7}{4}(0.5)^4(0.5)^3 = P(X=4)$

12. Let X and Y be two independent r.v.'s with X following Bernoulli($\frac{1}{2}$) and Y following Poisson(1). Then $P[XY = 0]$ is

A. $\frac{1}{2} + e^{-1}$

B. $\frac{1}{2} + \frac{e^{-1}}{2}$

C. $\frac{e^{-1}}{2}$

D. $\frac{1}{2}$

Solution:

$$\begin{aligned} P[XY=0] &= P[X=0] + P[Y=0] - P[X=0, Y=0] \\ &= P[X=0] + P[Y=0] - P[X=0]P[Y=0] \quad (X\, and\, Y \text{are independent}) \end{aligned}$$

Since X follows Bernoulli, $P[X=0] = q = \frac{1}{2}$.
As Y follows Poisson, $P[Y=0] = e^{-1}$. Therefore,

$$\begin{aligned} P[XY=0] &= \frac{1}{2} + e^{-1} - \frac{1}{2}e^{-1} \\ &= \frac{1}{2} + \frac{e^{-1}}{2} \end{aligned}$$

13. Which of the following statements is not true about a Poisson Probability Distribution with parameter λ?

A. The mean of the distribution is λ

B. The standard deviation of the distribution is the positive square root of λ

C. The parameter λ must be greater than zero

D. The parameter λ is coefficient of variation.

Solution: The coefficient of variation is $\frac{\sigma}{\mu} = \frac{\sqrt{\lambda}}{\lambda} = \frac{1}{\sqrt{\lambda}}$

14. Which of the following statements is not true about a Poisson Probability Distribution with parameter λ?

   A. The mean of the distribution is λ

   B. The variance of the distribution is λ

   C. The coefficient of variation is 1

   D. The parameter λ must be greater than zero .

   Solution: The coefficient of variation is $\frac{\sigma}{\mu} = \frac{\sqrt{\lambda}}{\lambda} = \frac{1}{\sqrt{\lambda}}$

15. A box contains N items with M defective items. A simple random sample of n items is selected without replacement and the number X of defective items in the sample is counted. Then the distribution of X is:

   A. Binomial distribution

   B. Discrete uniform distribution

   C. Negative binomial distribution

   D. Hypergeometric distribution

   Solution: Hypergeometric distribution.

16. Let X follow a Poisson(2) distribution. Then:

   A. The r.v. $2X$ follows Poisson(4) and $\frac{X}{2}$ follows Poisson(1)

   B. Both $2X$ and $\frac{X}{2}$ are not Poisson r.v.s.

   C. The r.v. $2X$ follows Poisson(4) but $\frac{X}{2}$ is not a Poisson r.v.

   D. The r.v. $2X$ is not a Poisson r.v. but $\frac{X}{2}$ is Poisson(1)

   Solution: Since $M_X(t) = e^{\lambda(e^t-1)}$,
   we have $M_{2X}(t) = M_X(2t) = e^{\lambda(e^{2t}-1)}$ and
   $M_{X/2}(t) = M_X(t/2) = e^{\lambda(e^{t/2}-1)}$.
   Hence both are not Poisson.

17. In a triangle test, a tester is presented with three fold samples, two of which are alike, and is asked to pick out the odd one by testing. If a tester has no well developed sense and can pick the odd one only, by chance, what is the probability that in five trials he will make four or more correct decisions?

   A. 1/243

   B. 10/243

   C. 233/243

   D. 11/243

   Solution: Here X follows Binomial(5,1/3).
   Hence $P[X \geq 4] = P[X = 4] + P[X = 5] = \binom{5}{4}(1/3)^4(2/3) + \binom{5}{5}(1/3)^5 = 11/243$

18. Experience has shown that a certain lie detector will show a positive reading (indicates a lie) 10% of the time when a person is telling truth. Suppose that a random sample of 4 suspects is subjected to a lie detector test regarding a recent one-person crime. Then the probability of observing reading if all suspects plead innocent and are telling the truth is:

    A. 0.4090

    B. 0.7390

    C. 0.6561

    D. 0.5905

    Solution: Here X follows Binomial(4,0.1).
    Hence $P(X=0)=(0.1)^0(0.9)^4=0.6561$.

19. Let X be a non-negative integer valued r.v. with mean equal to 1 and variance equal to 5/6. Then which of the following can be the distribution of X?

    A. Binomial with $n=6$ and $p=1/6$

    B. Poisson with $\lambda=1$

    C. Geometric with $p=1/6$

    D. Discrete uniform

    Solution:

    A. For Binomial, $mean=np=1$ and $variance=np(1-p)=5/6$, implies, $n=6$ and $p=1/6$.

    B. If X follows Poisson, then mean=variance, which is not true here.

    C. If X follows geometric with $p=1/6$, then $mean=1/p=6$, which is not true.

20. Suppose a die is tossed 5 times. What is the probability of getting exactly 2 fours?

    A. 0.161

    B. 0.171

    C. 0.250

    D. 0.333

    Solution: Here the random variable follows Binomial($5,\frac{1}{6}$).
    Hence the required probability is

$$P(X=2)=\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3=0.161$$

21. Bob is a high school basketball player. He is a 70% free throw shooter. That means his probability of making a free throw is 0.70. What is the probability that Bob makes his first throw on his fifth shot?

    (a) 0.0024

    (b) 0.0081

    (c) 0.0057

    (d) 0.1681

    Solution:

    His first free throw on his fifth shot means, first 4 shots are failed.

    Hence the required probability is $(0.3)^4 \times (0.7) = 0.0057$.

22. Let X be a r.v. with $B(n, p)$. Then the distribution of $n - X$ is

    (a) $B(n - 1, p)$

    (b) $B(n, 1 - p)$

    (c) $B(n - 1, q)$

    (d) $B(n, p)$

    Solution:

    If X follows $B(n, p)$, then $n - X$ follows $B(n, 1 - p)$ as if X represents success out of n with probability p, then $n - X$ represents failures out of n with probability $1 - p$.

23. Suppose that the probability that a patient with a certain disease will be cured by a certain treatment is 0.70. Suppose that the treatment is used on 50 patients with the disease. The expected value of the number of patients who are cured is

    (a) 7

    (b) 25

    (c) 35

    (d) 50

    Solution:

    Here X follows Binomial with $n = 50$ and $p = 0.70$.

    Hence the expectation, $E(X) = np = 50(0.70) = 35$.

24. Which of the following is not possible in case of Binomial distribution?

    (a) Mean/2=Variance

    (b) Mean<Variance

    (c) Mean>Variance

    (d) Trials are independent of each other.

Solution:
In Binomial, $mean = np$ and $variance = np - np^2$. Hence $variance \leq mean$.
So variance cannot be more than mean in Binomial.

25. Suppose that the probability that a cross between two varieties will express a particular gene is 0.20. What is the probability that in 8 progeny plants two or fewer plants will express the gene?

(a) 0.7969

(b) 0.6791

(c) 0.1678

(d) 0.7269

Solution:
Here $X \sim Binom(8, 0.20)$ and we have to find $P(X \leq 2)$.

$$\begin{aligned} P(X \leq 2) &= P(X = 2) + P(X = 1) + P(X = 0) \\ &= \binom{8}{2}(0.2)^2(0.8)^6 + \binom{8}{1}(0.2)(0.8)^7 + \binom{8}{0}(0.8)^8 \\ &= 0.6791 \end{aligned}$$

26. It is know that X follows Poisson distribution with mean λ and $P[X = 0] > P[X = 1]$. Then

(a) $\lambda = 1$

(b) $\lambda > 1$

(c) $\lambda < 1$

(d) We need more information to say anything about λ.

Solution:
If X follows Poisson, then $P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}$.
Therefore, $P[X = 0] > P[X = 1]$ implies $e^{-\lambda} < e^{-\lambda}\lambda$ and so $\lambda < 1$.

27. Let X denotes the number of independent throws of a fair die required to obtain the first occurrence of 3. Then $P[X \geq 6]$ is

(a) $(1/6)^5$

(b) $(1/6)^5(5/6)$

(c) $(5/6)^6$

(d) $(5/6)^5$

Solution:
Here X follows $Geometric(\frac{1}{6}$.
We know that $P[X \geq k] = (1 - p)^k$.
Hence $P[X \geq 6] = (5/6)^6$

28. The mean and variance of a binomial distribution are 8 and 4 respectively. Hence $P[X = 1]$ is equal to

    (a) $\frac{1}{2^{12}}$

    (b) $\frac{1}{2^4}$

    (c) $\frac{1}{2^6}$

    (d) $\frac{1}{2^8}$

    Solution:

    Given that $mean = np = 8$ and $variance = npq = 4$. So $q = 1/2$ and so $p = 1/2$.

    Therefore $n = 16$. Hence $P[X = 1] = \binom{16}{1}\left(\frac{1}{2}\right)^{16} = \frac{1}{2^{12}}$.

29. Let X be a Poisson random variable with $2P[X = 0] = P[X = 2]$. Then variance of X is given by

    (a) $\frac{1}{2}$

    (b) 1

    (c) 2

    (d) 3/2

    Solution:

    In Poisson distribution, $P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}$.

    So $2P[X = 0] = P[X = 2]$ implies $2e^{-\lambda} = e^{-\lambda}\lambda^2/2$ and so $\lambda = 2$.

    Therefore, $varaince = \lambda = 2$.

30. The number of accidents per week in a city has Poisson distribution with mean 3. What is the probability of exactly 2 accidents in 2 weeks?

    (a) $2e^{-3}$

    (b) $2e^{-6}$

    (c) e^{-6}

    (d) $18e^{-6}$

    Solution:

    Suppose X and Y represents number of accidents in week 1 and 2 , respectively.

    Then X and Y are independent Poisson random variables with $\lambda = 3$.

    And the probability of exactly 2 accidents in 2 weeks is

$$\begin{aligned} P[X=2,Y=0]+P[X=0,Y=2]+P[X=1,Y=1] &= P[X=2]P[Y=0]+P[X=0]P[Y=2]+P[X=1]P[Y=1] \\ &= 9e^{-6}+9e^{-6} \\ &= 18e^{-6} \end{aligned}$$

31. If the MGF of a binomial random variable is $M(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^5$, then the mean and variance are respectively

    (a) $(10/9, 5/3)$

(b) $(2/3, 1/3)$

(c) $(5/3, 10/9)$

(d) $(1/3, 5)$

Solution:

In binomial, $M(t) = (q + pe^t)^n = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^5$.

So we get $n = 5$, $p = 1/3$ and $q = 2/3$.

Hence $mean = np = 5/3$ and $variance = npq = 10/9$

32. The probability that a particular birth will be a male child is 0.52. Then the probability that out of 8 births there are two female births is

(a) $28(0.52)^6(0.48)^2$

(b) $28(0.52)^2(0.48)^6$

(c) $(0.52)^6(0.48)^2$

(d) $(0.52)^2(0.48)^6$

Solution:

Here $X \sim Binom(8, 0.52)$ and we want to find $P[X = 6]$.

So $P[X = 6] = \binom{8}{6}(0.52)^6(0.48)^2 = 28(0.52)^6(0.48)^2$

33. A box contains 10 white marbles and 15 black marbles. Ten marbles are drawn at random with replacement. Then the probability that x of these white in colour (x=0,1,2,...,10) is given by

(a) $\frac{\binom{10}{x}\binom{15}{10-x}}{\binom{25}{10}}$

(b) $\binom{10}{x}(\frac{2}{5})^x(\frac{3}{5})^{10-x}$

(c) $\binom{10}{x}(\frac{3}{5})^x(\frac{2}{5})^{10-x}$

(d) $\frac{\binom{10}{x}}{\binom{25}{10}}$

Solution:

Here $X \sim Binom(10, 2/5)$.

Hence $P[X = x] = \binom{10}{x}(\frac{2}{5})^x(\frac{3}{5})^{10-x}$

34. The moment generating function of Poisson distribution with parameter λ is

(a) $exp\{exp(\lambda t) - 1\}$

(b) $\lambda\{exp(t) - 1\}$

(c) $exp\{\lambda(1 - exp(t))\}$

(d) $exp\{\lambda(exp(t) - 1)\}$

Solution:

$M(t) = exp\{\lambda(exp(t) - 1)\}$.

