1 Sample space and Events

Definition 1.1. (Experiment) An *experiment* is the process whose outcome is not predictable with certainty in advance.

Example 1.2.
1. Tossing a coin once or several times
2. Tossing a 6 faced die.
3. Selecting a card or cards from a deck
4. Obtaining blood types from a group of individuals
5. Measuring (in hours) the lifetime of a transistor

Definition 1.3. (Sample Space) The set of all possible outcomes of an experiment is called the *sample space* of the experiment and is denoted by S.

Example 1.4.
1. If the experiment is tossing a coin, then the outcomes will be head and tail and hence the sample space is $S = \{H, T\}$.
2. If the experiment consists of flipping two coins, then the outcomes will be both heads, head and tail and both tails. So the sample space S consists of the following four points: $S = \{(H, H), (H, T), (T, H), (T, T)\}$.
3. If the experiment consists of tossing a die, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
4. If the experiment consists of tossing two dice, then the sample space consists of the 36 points $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$, where the outcome $(2, 5)$ means 2 appears in the first die and 5 on the other die.
5. If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers. That is $S = \{x \,|\, 0 \le x < \infty\}$

Definition 1.5. (Event) Any subset E of the sample space S is known as an *event*. An event is called *simple* or *elementary* if it consists of single outcome. If the outcome of the experiment is contained in E, then we say that the event E has occurred.

Example 1.6.
1. In the example 2, $E = \{(H, H), (H, T)\}$ is the event that a head appears on the first coin.
2. In example 4, $E_1 = \{(1, 5), (2, 4), (3, 3), (2, 4), (1, 5)\}$ is the event that the sum of the dice equals 6. The Event E_2 that the sum of dice equals 12 is the simple event as $E_2 = \{(6, 6)\}$ consists only one outcome.
3. In example 5 , $E = \{x \,|\, x \ge 10\}$ is the event that the transistor lasts at least 10 hours.

For any two events A and B of the sample space S, we can define the new events

- $A \cup B = \{\omega \in S \,|\, \omega \in A \,\text{or}\, \omega \in B\}$ consists of all outcomes that are either in A or in B or in A and B.
- $A \cap B = \{\omega \in S \,|\, \omega \in A \,\text{and}\, \omega \in B\}$ consists of all outcomes that are both in A and B.

- More generally, $\bigcup\limits_{i=1}^{\infty} A_i = \{\omega \in S \,|\, \omega \in A_i \text{ for some } i\}$ consists of all outcomes which lie in at least one event A_i and
  $\bigcap\limits_{i=1}^{\infty} A_i = \{\omega \in S \,|\, \omega \in A_i \text{ for all } i\}$ consists of all outcomes which lie in all events A_i
- $A^c = \{\omega \in S \,|\, \omega \notin A\}$ consists of all outcomes that are not in A.

Remark 1.7. • An event which doesn't contain any outcome is referred as *null event* and is denoted by $\emptyset$.

- Sample space is also an event referred as certain or sure event.

Definition 1.8. (Mutually Exclusive) Two events A and B of the sample space S are said to be *mutually exclusive* if $A \cap B = \emptyset$. i.e. if the event A occurs, then event B doesn't occur, and vive versa.

Example 1.9. If S is the sample space of tossing a die, then the events $A = \{\omega \in S \,|\, \omega \text{ is even}\}$ and $B = \{\omega \in S \,|\, \omega \text{ is odd}\}$ are mutually exclusive events.

Definition 1.10. Let A and B be two events of the sample space S. If $A \subset B$, then the occurrence of A necessarily implies the occurrence of B, i.e., the event B occurs whenever the event A occurs.

Theorem 1.11. *(DeMorgan's Laws) Let $A_1, A_2, ..., A_n$ are the events of the sample space S, then*

- $\left(\bigcup\limits_{i=1}^{n} A_i\right)^c = \bigcap\limits_{i=1}^{n} A_i^c$
- $\left(\bigcap\limits_{i=1}^{n} A_i\right)^c = \bigcup\limits_{i=1}^{n} A_i^c$

2 Axioms of Probability

Definition 2.1. Consider an experiment whose sample space is S. To each event A, a number $P(A)$, called the *probability of the event A* should satisfy the following, called the axioms of probability :

Axiom 1: $0 \leq P(A) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $A_1, A_2, ...,$
$P\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} P(A_i)$

Theorem 2.2. 1. *The probability of null event is zero, i.e. $P(\emptyset) = 0$*

2. *For any finite collection of mutually exclusive events $A_1, A_2, ..., A_n$,*
   $P\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} P(A_i)$
3. *for any event A,$P(A^C) = 1 - P(A)$*
4. *if $A \subset B then P(A) \leq P(B) and P(B \setminus A) = P(B) - P(A)$*
5. $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

Proof. 1. Consider the events $A_1 = S,\ A_i = \emptyset \quad \forall i \geq 2$. Then $A_i's$ are mutually exclusive events and hence by **Axiom 3**,
$P(S) = P(S) + \sum\limits_{i=2}^{n} P(\emptyset)$
$\Rightarrow \sum\limits_{i=2}^{n} P(\emptyset) = 0.$
$\Rightarrow P(\emptyset) = 0$ ($\because P(\emptyset) \geq 0$, by **Axiom 1**)

2. By defining $A_i = \emptyset$ for all $i \geq n+1$, we get

$$\begin{aligned} P\Big(\bigcup_{i=1}^{\infty} A_i\Big) &= \sum_{i=1}^{\infty} P(A_i) \\ \Rightarrow P\Big(\bigcup_{i=1}^{n} A_i\Big) &= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i) \\ &= \sum_{i=1}^{n} P(A_i) \quad (\because P(A_i) = 0\,\forall i \geq n+1) \end{aligned}$$

3. Since A and A^C are mutually exclusive events, we get
$P(A \cup A^c) = P(A) + P(A^c)$
$P(S) = P(A) + P(A^C)$
$1 = P(A) + P(A^C)$
$P(A^C) = 1 - P(A)$

4. If $A \subset B$, we can write $B = A \cup (B \setminus A)$
Since A and $B \setminus A$ are mutually exclusive, we have

$$\begin{aligned} P(B) &= P(A \cup (B \setminus A)) \\ &= P(A) + P(B \setminus A) \\ \therefore P(B \setminus A) &= P(B) - P(A) \end{aligned}$$

As $P(B \setminus A) \geq 0$, we get $P(B) \geq P(A)$.

5. We can write $A \cup B = A \cup \Big(B \setminus (A \cap B)\Big)$
Since A and $B \setminus (A \cap B)$ are disjoint, we get

$$\begin{aligned} P(A \cup B) &= P(A) + P(B \setminus (A \cap B)) \\ &= P(A) + P(B) - P(A \cap B) \quad (\because A \cap B \subset B) \end{aligned}$$

□

3 Finite Sample Space With Equally Likely Outcomes

Consider an experiment whose sample space S is a finite set, say $S = \{1, 2, 3, ..., n\}$.
Also suppose that all the outcomes of the sample space are equally likely to happen.
Then $P(\{1\}) = P(\{2\}) = ... = P(\{n\}) = k\,(say)$
$\Rightarrow P\Big(\{1\} \cup \{2\} \cup ... \cup \{n\}\Big) = P(S) = 1$

$\Rightarrow P(\{1\}) + P(\{2\}) + ... + P(\{n\}) = 1$

$\Rightarrow k + k + ... + k(n\, times) = 1$

$\Rightarrow nk = 1$

$\therefore P(\{i\}) = k = \frac{1}{n}$

Now suppose A is an event with m outcomes.

Let $A = \{a_1, a_2, ..., a_m\} \qquad 1 \leq a_i \leq n$. Then

$$\begin{aligned} P(A) &= P(\{a_1, a_2, ..., a_m\}) \\ &= P(\{a_1\}) + ... + P(\{a_m\}) \\ &= \frac{1}{n} + ... + \frac{1}{n}(m\, times) \\ &= \frac{m}{n} \\ P(A) &= \frac{No.\ of\ outcomes\ in\ A}{Total\ No.\ of\ outcomes} \\ &= \frac{No.\ of\ favourable\ outcomes\ for\ A}{Total\ No.\ of\ outcomes} \end{aligned}$$

Basic Principle of counting: Suppose that two experiment are to be performed. If experiment 1 can result in any one of m possible outcomes, and for each outcome from experiment 1, there are n possible outcomes of experiment 2, then together there mn possible outcomes of the two experiments.

Example 3.1. A class in probability theory consist of 6 men and 4 women. An exam is given and the students have ranked according to their performance. Assume that no two student obtain the same score.

1. How many different ranking are possible?
2. If all rankings are considered equally likely, what is the probability that women receive top four score?

Proof. 1. First rank by any of the 10 students and second rank by any of the remaining 9 student and third rank by remaining 8 and so on
Hence the total number of different ranking $= 10 \times 9 \times \cdots \times 1 = 10!$

2. The first rank can get by any one of 4 women and second rank by any one of the remaining 3 and so on and the fifth rank by any of 6 men and sixth rank by any one of remaining five men and so on.
The total number of favorable outcomes $4! \times 6!$
$\therefore$ The required probability$= \frac{4! \times 6!}{10!}$
$= \frac{4 \times 3 \times 2 \times 6!}{10 \times 9 \times 8 \times 7 \times 6!}$
$= \frac{1}{210}$.

□

Example 3.2. From a set of n items a random sample of size k is to be selected. What is the probability that a given item will be among the k selected?

Proof. Since there is one way of choosing the given item, the remaining $(k-1)$ items from the remaining $(n-1)$ items can be selected in ${}^{n-1}C_{k-1}$ ways.

Therefore the required probability $= \frac{\frac{(n-1)!}{(n-k)!(k-1)!}}{\frac{n!}{(n-k)!k!}}$
$= \frac{(n-1)!k(k-1)!}{n(n-1)!(k-1)!}$
$= \frac{k}{n}$

□

Example 3.3. If two dice are rolled. What is the probability that the sum of the up turned faces will equal to 7?

Proof. The sample space is $S = \{(i,j)|i,j = 1,2,...,6\}$ and $|S| = 36$. Let A be an event that represents the sum of upturned faces is 7, then

$$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$
$$\Rightarrow |A| = 6$$
$$\therefore P(A) = \frac{6}{36}$$
$$P(A) = \frac{1}{6}$$

□

Example 3.4. If three walls are randomly drawn from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and other two are black?

Proof. Let A be an event that represents 1 white and 2 black balls.
3 balls out of 11 balls can be drawn in $\binom{11}{3} = 165$ ways.
Now One white ball can be selected in 6 ways
and two black balls can be selected in $\binom{5}{2} = 10$ ways
$\therefore$ One white ball and two black balls can be selected in $6 * 10 = 60$ ways
$Hence\ P(A) = \frac{60}{165} = \frac{4}{11}$ □

Example 3.5. A committee of 5 is to a selected from a group of 6 en and 9 women. If a selection is made randomly, what is the probability that committee consist of 3 men and 2 women?

Proof. Let A be an event that represents 3 men and 2 women.
Since there are 15 members 5 members can be selected in $\binom{15}{5} = 3003$ ways
Now 3 men out off 6 men can be selected in $\binom{6}{3} = 20$ ways
and 2 women out off 9 women can be selected in $\binom{9}{2} = 36$ ways.

$$\therefore P(A) = \frac{20 \times 36}{3003}$$
$$= \frac{240}{1001}$$

□

4 NET/SET questions

1. Let E, F and G be three events such that $E \cap F = \emptyset$. Which of the following assignment of probabilities is not feasible?

   A. $P(E) = 0.2, P(F) = 0.8, P(G) = 0.3$

   B. $P(E) = 0.2, P(F) = 0.4, P(G) = 0.5$

   C. $P(E) = 0.2, P(F) = 0.8, P(G) = 0.4$

   D. $P(E) = 0.3, P(F) = 0.8, P(G) = 0.1$.

   Solution: Given that $E \cap F = \emptyset$. Implies $P(E \cap F) = 0$. So $P(E \cup F) = P(E) + P(F)$.
   If $P(E) = 0.3$ and $P(F) = 0.8$, then $P(E \cup F) = 1.1$, which is a contradiction.

2. Let E, F be two events such that $P(E) = 0.2$ and $P(F) = 0.8$. Which of the following statement is always true?

   A. $P(E \cup F) = 1$

   B. $P(E \cap F) \leq 0.2$

   C. $P(E \setminus F) = 0.6$

   D. $P(E \cap F) = 0.16$.

   Solution: As $E \cap F \subset F$, we get $P(E \cap F) \leq P(F) = 0.2$.

3. If event A occurs whenever event B occurs, then

   A. $P(B) \leq P(A)$

   B. $P(B) = P(A)$

   C. $P(A) = 1 - P(B)$

   D. $P(A) \leq P(B)$.

   Solution: It is given that the event A happens whenever event B occurs, so we have $B \subset A$, implies $P(B) \leq P(A)$.

4. A sample space consists of four points $\omega_1, \omega_2, \omega_3$, and ω_4. A valid assignment of probabilities to the points of this space is

   A. $P(\omega_1) = 0.1, P(\omega_2) = 0.2, P(\omega_3) = 0.6, P(\omega_4) = 0.1$

   B. $P(\omega_1) = 0.1, P(\omega_2) = 0.2, P(\omega_3) = 0.8, P(\omega_4) = 0.1$

   C. $P(\omega_1) = 0.1, P(\omega_2) = 0.3, P(\omega_3) = 0.3, P(\omega_4) = 0.1$

   D. $P(\omega_1) = 0.1, P(\omega_2) = 0.5, P(\omega_3) = 0.1, P(\omega_4) = 0.2$.

   Solution: As probability of sample space is 1, sum of these probabilities should be 1.

5. Let E and F be two events on the same probability space with sample space Ω. Suppose $P(E) = 0.4$ and $P(F) = 0.7$. Which of the following statements is always true?

   A. $E \cap F \neq \emptyset$

   B. $E \cup F = \Omega$

   C. $E \subset F$

   D. None of the above..

   Solution: If $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F) = 1.1$, which is not possible.
   So $E \cap F \neq \emptyset$.

6. E and F are events with $P(E) = 1/2 = P(F)$ and $P(E^c \cap F^c) = 1/3$. Then $P(E \cup F^c)$ is

   A. 1/4

   B. 5/6

   C. 2/3

   D. 3/4.

   Solution:

$$
\begin{aligned}
P(E \cup F^c) &= P(E) + P(F^c) - P(E \cap F^c) \\
&= P(E) + P\Big(F^c \setminus (E \cap F^c)\Big) \\
&= P(E) + P\Big(F^c \cap (E^c \cup F)\Big) \\
&= P(E) + P\Big(F^c \cap E^c\Big) \\
&= 1/2 + 1/3 = 5/6
\end{aligned}
$$

7. Let A and B be two events in a probability space. Suppose $P(A) = 0.3$ and $P(F) = 0.5$. Which of the following statements is always true?

   A. $P(A \cup B) = 0.8$

   B. $0.5 \leq P(A \cup B) \leq 0.8$

   C. $P(A \cup B) \geq 0.8$

   D. $P(A \cup B) = 0.65$..

   Solution: By using the result $max\{P(A), P(B)\} \leq P(A \cup B) \leq min\{P(A) + P(B), 1\}$, we get $0.5 \leq P(A \cup B) \leq 0.8$

8. Let E, F and G be mutually exclusive and exhaustive events. Then a feasible assignment of probabilities is

   A. $P(E) = 0.5$, $P(F) = 0.1$, $P(E \cap F) = 0.4$

   B. $P(E) = 0.8$, $P(E \cap F^c) = 0.2$

   C. $P(E) = 0.32$ and $P(F \cup G) = 0.68$

D. $P(E) = 0.32,\ P(F) = 0.4,\ P(E \cup G) = 0.22.$

Solution:

A. Since E and F are mutually exclusive, $E \cap F = \emptyset$. So $P(E \cap F) = 0$.

B. Since they are mutually exhaustive, $F \cup E \cup G = \Omega \Rightarrow F^c = E \cup G$.
So, $0.2 = P(E \cap F^c) = P(E \cap (E \cap G)) = P\Big((E \cap E) \cup (E \cap G)\Big) = P(E) = 0.8$, which is not possible.

C. As they are mutually exclusive, $F \cup G = E^c$. So $P(F \cup G) = P(E^c) = 1 - P(E) = 1 - 0.32 = 0.68$. This is feasible.

D. As $E \cup G = F^c$, $P(E \cup G) = P(F^c) = 1 - P(F) = 1 - 0.4 = 0.6$, which is not true.

9. If $P(E) = 1, P(F) = 0.3$, which of the following is not correct?

A. $P(E \cup F) = 1,\ P(E \cap F) = 0.3$

B. $P(E^c \cup F) = 0.3,\ P(E \cap F) = 0.3$

C. $P(E \cup F^c) = 1,\ P(E \cap F^c) = 0.4$

D. $P(E \cup F) = 1,\ P(E^c \cap F) = 0.$

Solution: If $P(E \cup F^c) = 1$, then $P(E) + P(F^c) - P(E \cap F^c) = 1$.
Implies $P(E \cap F^c) = 0.7$. So (C) is not correct.

10. Which of the following cases has $s = \{a, b, c\}$ constitute a probability space?

A. $P(a) = 0.5, P(b) = 0.2, P(c) = 0.3$

B. $P(a) = 0.5, P(b) = 0.3, P(c) = 0.3$

C. $P(a) = 0.5, P(b) = 0.7, P(c) = -0.2$

D. $P(a) = 0.5, P(b) = 0.2, P(c) = 0.2.$

Solution:
As $P(s) = 1$, we should have $P(a) + P(b) + P(c) = 1$. Also probability of any event is non negative, so A constitute the probability space.

11. An urn contains 6 black and 5 white balls. Two balls are drawn one by one at random with replacement. The probability of getting two white balls is

A. 2/11

B. 4/25

C. 25/121

D. 2/25.

Solution: Total number of ways of choosing 2 balls out of 11 balls *(with replacement)* is $11 \times 11 = 121$
The number of ways that the two chosen balls *(with replacement)* are white is $5 \times 5 = 25$.
Hence the required probability is 25/121.

12. An urn contains 4 red, 8 green and 2 yellow balls. 5 balls are randomly selected with replacement. What is the probability that 1 red, 2 green, and 2 yellow balls will be selected?

    A. $\frac{32}{7^5}$

    B. $\frac{1920}{7^5}$

    C. $\frac{960}{7^5}$

    D. $\frac{15}{512}$.

    Solution: Total number of ways of choosing 5 balls out of 14 balls *(with replacement)* is $(14)^5$
    Number of ways of selecting 1 red, 2 green, and 2 yellow balls *(with replacement)* is $4 \times 8^2 \times 2^2 = 2^{10}$
    Hence the required probability is $\frac{2^{10}}{2^5 \times 7^5} = \frac{32}{7^5}$.

13. A card is drawn randomly from a deck of ordinary playing cards. You win Rs.10 if the card is a spade or an ace. What is the probability that you will win the game?

    A. 1/13

    B. 13/52

    C. 17/52

    D. 4/13.

    Solution: Let A be an event that represents the selected card is a spade and B be an event that it is an ace. Then the required probability is

    $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$

    ($\because$ Out of 52 cards, there are 13 spade cards; 4 aces; and 1 spade which is also an ace)

14. The total number of samples of size 2 that can be drawn with replacement from a population of 10 units is

    A. 10^2

    B. 2^{10}

    C. 90

    D. 45.

    Solution: Number of ways of 2 units from 10 units with replacement is $10 \times 10 = 10^2$.

15. An urn containing 9 balls 2 of which are red, 3 are blue and 4 are black balls. When 3 balls are drawn at random, the chance that they are of the same colour is

    A. 5/84

    B. 3/9

    C. 3/7

    D. 7/17.

Solution: The total number of ways of selecting 3 balls out of 9 is $\binom{9}{3} = 84$. Now the selected balls are of same colour means they must be all blue or all black. If all are blue, this can be done in 1 way, and if all are black, this can be done in $\binom{4}{3} = 4$ ways. So there are totally 5 ways to select the same colours balls.
Hence the required probability is 5/84.

16. A sample space consists of five simple events E_1, E_2, E_3, E_4 and E_5. If $P(E_1) = P(E_2) = 0.1, P(E_3) = 0.4$ and $P(E_4) = 3P(E_5)$. Then $P(E_4)$ and $P(E_5)$:

A. are 0.06 and 0.02 respectively

B. cannot be determined from the given information

C. are 0.6 and 0.2 respectively

D. are 0.3 and 0.1 respectively.

Solution:
As given events are simple, we get that
$P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) = P(\Omega) = 1$
$\Rightarrow 0.1 + 0.1 + 0.4 + 3P(E_5) + P(E_5) = 1 \Rightarrow P(E_5) = 0.1$ and $P(E_4) = 0.3$.

5 Conditional Probability

To understand the conditional probability, consider the following example. Let A be the event that denotes the sum of the faces of the two dice is 8. Then $P(A) = \frac{5}{36}$ as there are 5 favorable outcomes $(2, 6), (3, 5), (4, 4), (5, 3)$ and $(6, 2)$, out of 36 possible outcomes.
Consider another event B which represents the face of the first die is 3. Then $P(B) = \frac{6}{36} = \frac{1}{6}$, as there are 6 favorable outcomes namely $(3, 1), (3, 2), (3, 3), (3, 4), (3, 5)$ and $(3, 6)$, out of 36 possible outcomes.
Now suppose that the event B occurred, i.e. the first die is 3. Then what is the probability of A given event B occurred?
The assumption that the event B took place reduces the possible outcomes to 6. Among these outcomes, there is only one favorable outcome, namely $(3, 5)$. Hence the probability of A given B is $\frac{1}{6}$ and is denoted by $P(A|B)$, called the conditional probability of A given B.
Therefore,

$$\begin{aligned} P(A|B) &= \frac{\text{No. of favourable outcomes to A given B}}{\text{Total outcomes given B}} \\ &= \frac{\text{No. of favourable outcomes to } A \cap B}{\text{No. of favourable outcomes to B}} \\ &= \frac{\frac{\text{No. of favourable outcomes to } A\cap B}{\text{Total possible outcomes}}}{\frac{\text{No. of favourable outcomes to } A\cap B}{\text{Total possible outcomes}}} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

Definition 5.1. The conditional probability of A given B is denoted by $P(A|B)$ and is given by
$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) > 0.$

Example 5.2. If a coin is flipped twice and if we assume that all four points in sample space $S = (H,H)(H,T)(T,H)(T,T)$ are equally likely, what is the conditional probability that both flips result in head. Given that first flip does? (that is first is head).

Proof. Let A represents both heads and B represents first is head.
Then $P(A \cap B) = \frac{1}{4}$, $P(B) = \frac{1}{2}$.
Hence $P(A|B) = \frac{1}{2}$. □

Multiplication theorem: If $P(A), P(B) > 0$, then the probability of both A and B occurs is $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$.

Theorem 5.3 (Baye's Theorem)**.** *If the events $A_1, A_2, ..., A_k$ constitute a partition of the sample space S such that $P(A_i) \neq 0 \forall i = 1, 2, ..., k$, then for any event A of S such that $p(A) \neq 0$,*
$P(A_r|A) = \frac{P(A_r)P(A/A_r)}{\sum_{i=1}^{k} P(A_i)P(A|A_i)}$

Proof.

$$\begin{aligned} P(A_r|A) &= \frac{P(A_r \cap A)}{P(A)} \\ &= \frac{P(A_r)P(A|A_r)}{P(A)} \end{aligned}$$

We show that $P(A) = \sum_{i=1}^{k} P(A_i)P(A|A_i)$.
Since $S = \bigcup_{i=1}^{k} A_i$, and $A = A \cap S$, we get

$$\begin{aligned} A &= A \cap (\bigcup_{i=1}^{k} A_i) \\ &= \bigcup_{i=1}^{k} (A \cap A_i) \\ \Rightarrow P(A) &= \sum_{i=1}^{k} P(A \cap A_i) \\ \Rightarrow P(A) &= \sum_{i=1}^{k} P(A_i)P(A|A_i) \end{aligned}$$

Therefore, $P(A_r|A) = \frac{P(A_r)P(A/A_r)}{\sum_{i=1}^{k} P(A_i)P(A|A_i)}$. □

Example 5.4. Urn A contains 10 black and 10 white balls. Urn B contains 8 black and 10 white balls. One urn is chosen at random from A and B and a ball is chosen at random from the chosen urn, which happens to be a white ball. Then what is the probability that chosen urn is A?

Proof. Let A_1 be the event of chosen urn is A, A_2 be the event of chosen urn is B and A_3 be the event of chosen ball is white.
Then $P(A_1) = P(A_2) = \frac{1}{2}$
We have to find $P(A_1|A_3)$. By Baye's theorem,

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{\frac{1}{2}\frac{10}{20}}{\frac{1}{2}\frac{10}{20} + \frac{1}{2}\frac{10}{18}}$$
$$= \frac{9}{19}$$

□

Example 5.5. We have two identical urns A and B such that A contains 3 red and 2 black balls while B contains 3 black and 3 red balls. One urn is chosen at random and balls are repeatedly drawn from the urn with replacement. If three drawn resulted in red coloured ball, what is the probability that urn A was chosen?

Proof. Let A_1 be the event of chosen urn is A, A_2 be the event of chosen urn is B and A_3 be the event of chosen balls are red coloured.
Then $P(A_1) = P(A_2) = \frac{1}{2}$
We have to find $P(A_1|A_3)$. By Baye's theorem,

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$
$$= \frac{\frac{1}{2}\frac{3^3}{5^3}}{\frac{1}{2}\frac{3^3}{5^3} + \frac{1}{2}\frac{3^3}{5^3}}$$
$$= \frac{216}{341}$$

□

Example 5.6. A popular car comes in both a petrol and diesel version. Each of these further available in 3 models, L,V and Z. Among all owners of the petrol version of this car, 50% have model V and 20% have model Z. Among diesel car customers, 50% have model L and 20% model V.60% of all customers have brought diesel cars. If a randomly chosen customer has model V, what is the probability that the car is a diesel car?

Proof. Let A_1 be the event of chosen car is petrol, A_2 be the event of chosen car is diesel and A_3 be the event of chosen car is of model V.
Then $P(A_1) = 0.4, P(A_2) = 0.6, P(A_3|A_1) = 0.5$ and $P(A_3|A_2) = 0.2$
We have to find $P(A_2|A_3)$. By Baye's theorem,

$$P(A_2|A_3) = \frac{P(A_2)P(A_3|A_2)}{P(A_1)P(A_3|A_1) + P(A_2)P(A_3|A_2)}$$
$$= \frac{0.6 \times 0.2}{0.4 \times 0.5 + 0.6 \times 0.2}$$
$$= \frac{3}{8}$$

□

6 Independent Events

Two events A and B are said to be independent events if the occurrence of one of them does not affect the probability of the occurrence of the other. Clearly, if $P(A) = 0$ or $P(B) = 0$, then A and B are independent events as the chance of occurrence is zero, it doesn't affect the probability of the other. So we can assume $P(A) > 0$ and $P(B) > 0$, when we talk about independent events and the mathematical definition is the following.

Definition 6.1 (Independent Events)**.** Let A and B be two events with positive probability. Then A and B are said to be independent events if the occurrence of one of them does not affect the probability of the occurrence of the other, i.e. $P(A|B) = P(A)$ or $P(B|A) = P(B)$. Otherwise, they are dependent events.

Remark 6.2. If A and B are independent, then $P(A|B) = P(A)$ implies $P(A \cap B) = P(A)P(B)$ and this identity is valid even if $P(A) = 0$ or $P(B) = 0$ as $P(A \cap B) \leq \min\{P(A), P(B)\}$. Hence A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

Example 6.3.

1. If $P(A) = 0$ or $P(A) = 1$, then A is independent to every event.
2. If A is independent to itself, then $P(A) = 0$ or $P(A) = 1$.

Proof.

1. Let B be any event.
   To show A and B are independent, we show that $P(A \cap B) = P(A)P(B)$.
   Suppose $P(A) = 0$. Then $P(A \cap B) \leq P(A) = 0$ implies $P(A \cap B) = 0 = P(A)P(B)$.
   Suppose $P(A) = 1$. Then $P(A^c) = 0$ and so $P(A^c \cap B) = 0$.
   As $P(B) = P(A \cap B) + P(A^c \cap B)$, we get $P(B) = P(A \cap B)$.
   Therefore, $P(B) = P(B)P(A) = P(A \cap B)$.
2. If A is independent to itself, then $P(A) = P(A \cap A) = P(A)P(A)$.
   Hence $P(A) = 0$ or $P(A) = 1$.

□

Theorem 6.4. *If A and B are independent events, then*

1. *A and B^c are independent*
2. *A^c and B are independent*
3. *A^c and B^c are independent.*

Proof.

1. Since $P(A \cap B^c) = P(A \setminus (A \cap B))$, we get

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B)) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c). \end{aligned}$$

2. Since $P(B \cap A^c) = P(B \setminus (A \cap B))$, we get

$$\begin{aligned} P(B \cap A^c) &= P(B) - P(A \cap B)) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c). \end{aligned}$$

3. A^c and B^c are independent.

□

7 NET/SET questions

1. If an event B is independent of itself, then $P(B)$ is:

   A. 0

   B. 1

   C. 1 0r 0.5

   D. 0 or 1

   Solution: If B is independent to itself, then
   $P(B) = P(B \cap B) = P(B)P(B) \Rightarrow P(B)(1 - P(B)) = 0.$
   Hence $P(B) = 0$ or $P(B) = 1$.

2. It is given that $P(A|B) = 0.4$ and $P(A|B^c) = 0.6$, then

   A. $0 \leq P(A) \leq 0.4$

   B. $P(A) = 0.5$

   C. $0.6 \leq P(A) \leq 1$

   D. $0.4 \leq P(A) \leq 0.6$

   Solution:
   $P(A|B) = 0.4 \Rightarrow P(A \cap B) = 0.4(P(B))$
   $P(A|B^c) = 0.6 \Rightarrow P(A \cap B^c) = 0.6(P(B^c)) = 0.6 - 0.6(P(B))$
   Now $P(A) = P(A \cap B^c) + P(A \cap B) = 0.6 - 0.6(P(B)) + 0.4(P(B)) = 0.6 - 0.2(P(B))$
   So $P(A) = 0.6 - 0.2(P(B)) \leq 0.6$
   And $P(B) \leq 1 \Rightarrow P(A) \geq 0.6 - 0.2 = 0.4.$
   Therefore, $0.4 \leq P(A) \leq 0.6$

3. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $P(\omega) = \frac{1}{6}$ for all $\omega \in \Omega$.
   Let $A_1 = \{\omega \,|\, \omega \text{ is divisible by } 2\}$ and $A_2 = \{\omega \,|\, \omega \text{ is divisible by } 3\}$. Then

   A. A_1 and A_2 are mutually exclusive

   B. $A_1 \cup A_2 = \Omega$

   C. A_1 and A_2 are independent

   D. None of these.

   Solution: Here $A_1 = \{2, 4, 6\}$, $A_2 = \{3, 6\}$ and $A_1 \cap A_2 = \{6\}$.
   So, $P(A_1) = \frac{3}{6} = \frac{1}{2}$, $P(A_2) = \frac{2}{6} = \frac{1}{3}$ and $P(A_1 \cap A_2) = \frac{1}{6}$.
   Therefore, $P(A_1 \cap A_2) = P(A_1)P(A_2)$ implies they are independent.

4. Let E, F and G be mutually independent events. Which of the following statement is not true?

   A. $P(E|F) = P(E|F^c)$

B. The events $E \cap F$ and G are independent

C. The events $E \cup F$ and G^c are not independent

D. The events $E \cap F$ and $F \cap G$ are dependent.

Solution:

A. If E and F are independent then E and F^c are also independent.
So $P(E|F) = P(E|F^c)$.

B. $P\Big((E \cap F) \cap G\Big) = P(E \cap F \cap G) = P(E)P(F)P(G) = P(E \cap F)P(G)$. So they are independent.

C.

$$\begin{aligned} P\Big((E \cup F) \cap G^c\Big) &= P\Big((E \cap G^c) \cup (F \cap G^c)\Big) \\ &= P(E \cap G^c) + P(F \cap G^c) - P(E \cap G^c \cap F \cap G^c) \\ &= P(E)P(G^c) + P(F)P(G^c) - P(E)P(F)P(G^c) \\ &= \Big(P(E) + P(F) - P(E \cap F)\Big)P(G^c) \\ &= P(E \cup F)P(G^c) \end{aligned}$$

So they are independent.

D. $P\Big((E \cap F) \cap (F \cap G)\Big) = P(E \cap F \cap G) = P(E)P(F)P(G)$
where as $P(E \cap F)P(F \cap G) = P(E)P(F)P(F)P(G)$. So they are dependent.

5. Let E, F and G be three independent events with $P(E) = 0.2, P(F) = 0.1$ and $P(G) = 0.9$. Which of the following statement is incorrect?

A. $P(E^c \cap G) = 0.72$ and $P(E \cap G) = 0.18$

B. $P(E \cup F) = 0.28$ and $P(F \cap G) = 0.09$

C. $P(E \cap F) = 0.02$ and $P(E \cap F \cap G) = 0.018$

D. $P(E \cap F \cap G) = 0.018$ and $P(E \cup G) = 1$.

Solution: $P(E \cup G) = P(E) + P(G) - P(E)P(G) = 0.2 + 0.9 - 0.18 = 0.92$.

6. Let E and F be two events on a probability space (Ω, F, P). Which of the following statement is not always true?

A. $P(E \setminus F) = P(E) - P(F)$

B. $P(E \cup F) \geq P(E) - P(E \cap F)$

C. $P(E^c \cap F^c) = P(E^c) - P(F \cap E^c)$

D. $P(E|F) = 1 - P(E^c|F)$.

Solution:

A. $P(E \setminus F) = P(E) - P(F)$ is true only if $F \subset E$.

B. $P(E \cup F) = P(E) + P(F) - P(E \cap F) \geq P(E) - P(E \cap F)$ as $P(F) \geq 0$.

C.

$$\begin{aligned} P(E^c \cap F^c) &= P\big((E \cup F)^c\big) \\ &= 1 - P(E \cup F) \\ &= 1 - [P(E) + P(F) - P(E \cap F)) \\ &= 1 - P(E) - P(F) + P(E \cap F) \\ &= P(E^c) - [P(F) - P(E \cap F)] \\ &= P(E^c) - P\Big(F \setminus (E \cap F)\Big) \qquad (\because E \cap F \subset F) \\ &= P(E^c) - P(F \cap E^c) \qquad (\because F \setminus (E \cap F) = F \cap (E^c \cup F^c) = F \cap E^c) \end{aligned}$$

D.

$$\begin{aligned} 1 - P(E^c|F) &= 1 - \frac{P(E^c \cap F)}{P(F)} \\ &= \frac{P(F) - P(E^c \cap F)}{P(F)} \\ &= \frac{P(F \setminus (E^c \cap F))}{P(F)} \\ &= \frac{P(F \cap (E \cup F^c))}{P(F)} \\ &= \frac{P(F \cap E)}{P(F)} \\ &= P(E|F) \end{aligned}$$

7. Let a sample space consists of four points $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ and suppose the assigned probabilities are $P(\omega_1) = P(\omega_2) = 0.2$ and $P(\omega_3) = P(\omega_4) = 0.3$. Which of the following statement is correct?

   A. Let $A = \{\omega_1, \omega_2, \omega_3\}$ and $B = \{\omega_3\}$. Then $P(B|A) = 1$

   B. The events $\{\omega_1, \omega_2\}$ and $\{\omega_2, \omega_4\}$ are dependent

   C. Let $A = \{\omega_1, \omega_2, \omega_3\}$ and $B = \{\omega_3\}$. Then $P(A|B) = 1$

   D. The events $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ are independent.

   Solution:

   A. Here $P(A) = P(\omega_1) + P(\omega_2) + P(\omega_3) = 0.7$ and $P(A \cap B) = P(\omega_3) = 0.3$.
   So $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3}{7}$

   B. $P(\{\omega_1, \omega_2\}) = 0.4$ and $P(\{\omega_2, \omega_4\}) = 0.5$.
   So $P(\{\omega_1, \omega_2\})P(\{\omega_2, \omega_4\}) = 0.2 = P\Big(\{\omega_1, \omega_2\} \cap \{\omega_2, \omega_4\}\Big) = P(\omega_2) = 0.2$
   Hence they are independent.

   C. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\omega_3)}{P(\omega_3)} = 1$

   D. $P(\{\omega_1, \omega_2\}) = 0.4$ and $P(\{\omega_3, \omega_4\}) = 0.6$.
   So $P(\{\omega_1, \omega_2\})P(\{\omega_2, \omega_4\}) = 0.24$

Where as $P\Big(\{\omega_1,\omega_2\}\cap\{\omega_3,\omega_4\}\Big) = P(\emptyset) = 0$
Hence they are dependent.

8. Let A and B be two disjoint events with $0 < P(A) < 1$ and $0 < P(B) < 1$. Which of the following is always true?

   A. $P(A \cup B) = 1$

   B. A and B are dependent events

   C. $P(A) + P(B) = 1$

   D. $P(A|B) = P(A)$.

   Solution: If A and B are disjoint, then $P(A\cap B) = 0$, but $P(A)P(B) \neq 0$ ($\because$ both are positive). So they are dependent events.

9. Let A and B be two independent events with $P(A) > 0$ and $P(B) > 0$. Which of the following statement is false?

   A. $P(A|B) = P(A)$

   B. If $A \subset B$, then $P(A \cup B) = 1$

   C. $P(A^c \cap B^c) = 0$

   D. If $A \subset B$, then $P(A^c \cap B) = P(A^c)$.

   Solution: Given that $P(A\cap B) = P(A)P(B)$ and $P(A), P(B)$ are positive.

   A. $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$.

   B. If $A \subset B$, then $A \cap B = A$ and $A \cup B = B$. So $P(A) = P(A\cap B) = P(A)P(B)$.
   Since $P(A) > 0$, we get that $P(B) = P(A \cup B) = 1$.

   C. Since A and B are independent, we get that A^c and B^c are independent.
   So $P(A^c \cap B^c) = P(A^c)P(B^c)$. This need not be zero always.

   D. If $A \subset B$, by (B), $P(B) = 1$
   So $P(A^c \cap B) = P(A^c)P(B) = P(A^c)$.

10. Let E and F be two independent and mutually exclusive events. Which of the following statements is not correct?

   A. Either $P(E) = 1$ or $P(F) = 1$

   B. Either $P(E) = 0$ or $P(F) = 0$

   C. $P(E) = P(E \cup F) - P(F)$

   D. $P(E|F) = P(F)$.

Solution: Given that E and F are two independent and mutually exclusive. So $E \cap F = \emptyset$ and $P(E \cap F) = P(E)P(F)$.

Hence, $P(E)P(F) = P(\emptyset) = 0$. Implies $P(E) = 0$ or $P(F) = 0$.

Also $P(E \cup F) = P(E) + P(F)$ implies $P(E) = P(E \cup F) - P(F)$

$P(E|F) = 0 \neq P(F)$. Hence answer is (D).

11. A and B are events such that $P(A|B) = P(A|B^c)$. Then we can conclude that

A. A and B are disjoint

B. A and B are independent

C. $P(B|A) = P(B|A^c)$

D. $P(A) \neq P(B)$.

Solution:

$$\begin{aligned}
\frac{P(A \cap B)}{P(B)} &= \frac{P(A \cap B^c)}{P(B^c)} \\
P(A \cap B)(1 - P(B)) &= P(B)P(A \cap B^c) \\
P(A \cap B) &= [P(A \cap B) + P(A \cap B^c)]P(B) \\
&= P\Big((A \cap B) \cup (A \cap B^c)\Big)P(B) \\
&= P(A \cap (B \cup B^c))P(B) \\
&= P(A \cap \Omega)P(B) \\
&= P(A)P(B)
\end{aligned}$$

Hence A and B are independent.

12. For any events A and B, which of the following are always true?

A. $P(A \,\text{or}\, B) = P(A) + P(B)$

B. $P(A \,\text{or}\, B) = P(A) + P(B) - P(A \,\text{and}\, B)$

C. $P(A \,\text{or}\, B) = P(A).P(B)$

D. $P(A \,\text{and}\, B) = P(A).P(B)$.

Solution: Given that $P(A \cap B) = P(A)P(B)$ and $P(A), P(B)$ are positive.

A. This is true only if A and B are mutually exclusive, i.e. $A \cap B = \emptyset$

B. This is always true.

C. not true, in fact no such relation exists.

D. True if A and B are independent.